

Tile Number and Space-Efficient Knot Mosaics

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Abstract

In this paper we introduce the concept of a space-efficient knot mosaic. That is, we seek to determine how to create knot mosaics using the least number of non-blank tiles necessary to depict the knot. This least number is called the tile number of the knot. We provide a complete list of every prime knot with mosaic number six or less, including a minimal, space-efficient knot mosaic for each of these. We also determine the tile number or minimal mosaic tile number of each of these prime knots.

1 Introduction

Mosaic knot theory is a branch of knot theory that was first introduced by Kauffman and Lomonaco in the paper *Quantum Knots and Mosaics* [4] and was later proven to be equivalent to tame knot theory by Kuriya and Shehab in the paper *The Lomonaco-Kauffman Conjecture* [1]. This approach involves creating a knot mosaic by sectioning off a standard knot diagram into an $n \times n$ array of *mosaic tiles* selected from the collection of eleven tiles shown in Figure 1. Each arc and crossing of the original knot projection is represented by arcs, line segments, or crossings drawn on each tile. These tiles are identified, respectively, as $T_0, T_1, T_2, \dots, T_{10}$. Tile T_0 is a blank tile, and we refer to the rest collectively as non-blank tiles.

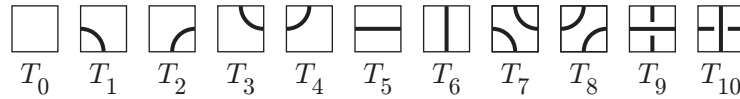


Figure 1: Tiles $T_0 - T_{10}$.

In order to define a knot mosaic we introduce a few simple terms. A *connection point* of a tile is a midpoint of a tile edge that is also the endpoint of a curve drawn on the tile. Two tiles are *contiguous* if they lie immediately next to each other in either the same row or the same column. A tile is *suitably connected* if each of its connection points touches a connection point of a contiguous tile. Two tiles are *diagonally adjacent* if they share two contiguous tiles, that is, their array position differs by exactly one row and one column.

Definition. An $n \times n$ array of tiles is an $n \times n$ *knot mosaic*, or n -*mosaic* if each of its tiles are suitably connected.

Note that an n -mosaic could represent a knot or a link, as illustrated in Figure 2. The first two mosaics depicted are 4-mosaics, and the third one is a 5-mosaic.

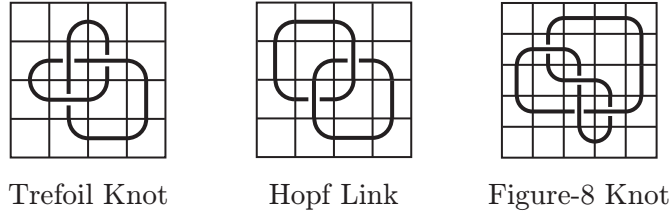


Figure 2: Examples of knot mosaics.

One particular piece of information of interest is the invariant known as the mosaic number of a knot or link. The *mosaic number* of a knot or link K is the smallest integer n for which K can be represented as an n -mosaic. We denote the mosaic number of K as $m(K)$.

Finding bounds on the mosaic number in terms of the crossing number of the knot or link has been a primary focus of research in mosaic knot theory. Lee, Hong, Lee, and Oh, in their paper *Mosaic Number of Knots* [2], found an upper bound for the mosaic number m for a knot or link with crossing number c . In particular, for any nontrivial knots and non-split links other than the Hopf link, $m \leq c + 1$. In the case of a prime, non-alternating link (except the 6^3_3 link), they show that $m \leq c - 1$.

The mosaic number has been determined for every prime knot with crossing number 8 or less. For details, see *Knot Mosaic Tabulations* [3] by Lee, Ludwig, Paat, and Peiffer. In particular, the mosaic number of the unknot is 2, the mosaic number of the trefoil knot is 4, and the mosaic number of the figure-8 knot (among others) is 5. Every prime knot with eight crossings or less has mosaic number at most 6. In this paper, we determine all prime knots that have mosaic number at most 6.

As we work with knot mosaic diagrams, we can move parts of the knot around within the mosaic via planar isotopy moves, similar to how we use planar isotopy moves to alter standard knot diagrams. There are a number of *mosaic planar isotopy moves* that are analogous to the planar isotopy moves for standard knot diagrams. There are also *mosaic Reidemeister moves* that are analogous to the standard Reidemeister moves for standard knot diagrams. A complete list of all of these moves are given and discussed in [4] and [1]. In this paper, we will simply refer to these as planar isotopy moves.

We also point out that throughout this paper we make significant use of the software package KnotScape [5], created by Thistlethwaite, to verify that a given knot mosaic represents a specific knot. Without this program, the authors of this paper would not have been able to complete the work.

2 Space-Efficient Knot Mosaics

Any given knot can be represented as a knot mosaic in many different ways, perhaps on a mosaic that is larger than necessary, perhaps with unnecessary, complicating features such as loops, bends or twists, or perhaps with unnecessary “empty space” causing the mosaic to have more non-blank tiles than absolutely necessary. In this paper, we want to explore, in some sense, the “most efficient way” to represent a knot as a knot mosaic. A few examples of the trefoil knot are given in Figure 3. It seems clear that the middle two knot mosaics are not represented in an overly efficient way. One has a simple but unnecessary loop that can be removed via a Reidemeister Type I move, and the other can be simplified by rotating the lowest crossing clockwise and shifting it up one row. Both are on a larger-than-necessary 5×5 mosaic. The first and last knot mosaics in Figure 3 are similar to each other, but the first uses thirteen non-blank tiles and the last uses only twelve non-blank tiles. Of the four mosaics depicted, the last one uses the least amount of space within the smallest possible mosaic, and this is the type of “efficiency” we want to explore.

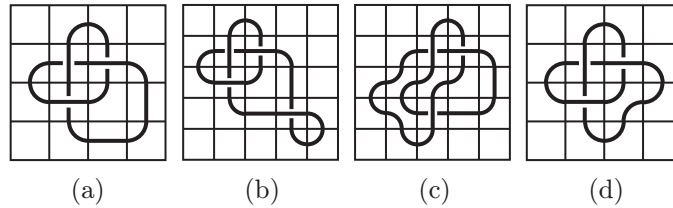


Figure 3: Examples of trefoil knot mosaics.

Definition. A knot mosaic is called *minimal* if it is a realization of the mosaic number of the knot. That is, a knot with mosaic number n is depicted as an n -mosaic.

Definition. A knot mosaic is called *reduced* if there are no unnecessary, reducible crossings in the knot mosaic diagram. That is, we cannot draw a simple, closed curve on the knot mosaic that intersects the knot diagram transversely at a single crossing but does not intersect the knot diagram at any other point.

Definition. The *tile number of a mosaic* is the number of non-blank tiles (all tiles except T_0) used to create that specific mosaic.

Definition. The *tile number* $t(K)$ of a knot or link K is the fewest non-blank tiles needed to construct K . That is, it is the smallest possible tile number of all possible mosaic diagrams for K .

Definition. The *minimal mosaic tile number* $t_m(K)$ of a knot or link K is the fewest non-blank tiles needed to construct K on a minimal mosaic. That is, it is the smallest possible tile number of all possible minimal mosaic diagrams for K .

Note that in this last definition we are intentionally coupling the tile number of a knot with the mosaic number of the knot. That is, for a knot with mosaic number n , the minimal mosaic tile number of the knot is the least number of non-blank tiles needed to construct the knot on an $n \times n$ mosaic. Much like the crossing number of a knot cannot always be realized on a minimal mosaic (such as the 6_1 knot), the tile number of a knot cannot always be realized on a minimal mosaic, which is why we distinguish between tile number and minimal mosaic tile number. Note that the tile number of a knot or link K is certainly less than or equal to the minimal mosaic tile number of K , $t(K) \leq t_m(K)$. The fact that the tile number of a knot is not necessarily equal to the minimal mosaic tile number of the knot is confirmed later in Theorem 26. However, for prime knots, if the mosaic number of the knot is 5 or less, the tile number and minimal mosaic tile number are equal. We will formalize this in Corollary 17. Just to be clear, when referring to a mosaic, we simply have the tile number. When referring to a knot, we must distinguish between tile number or minimal mosaic tile number.

As we have seen in the example mosaics in Figure 3, there are multiple ways to represent a given knot or link on a mosaic. Some knot mosaics are minimal, as in Figures 3(a) and (d), and some are not minimal, as in Figures 3(b) and (c). Some knot mosaics are reduced, and some are not. Figure 3(b) is not reduced. There are also some knot mosaics that use more tiles than necessary, even if they are reduced and minimal. The tile number of each of the knot mosaics in Figure 3 are 13, 16, 18, and 12, respectively. Figure 3(d) uses the smallest number of non-blank tiles among the given mosaics.

In some sense, it seems that if we push the arcs of the knot mosaic inward as much as possible, we can decrease the number of non-blank tiles used to create the mosaic. Similarly, if we can cluster the crossings together as much as possible, then the number of other tiles needed to connect the crossing tiles to each other could possibly be decreased. We can formalize this notion with the following definitions. The first definition does not allow for moving the crossings within the mosaic, but the second definition does.

Definition. A knot mosaic is *diagram space-efficient* if its tile number has been minimized through a series of planar isotopy moves that do not change the location or type of crossing tiles. That is, there is no sequence of planar isotopy moves that reduces the tile number of the mosaic without changing the location or type of the crossings in that mosaic knot diagram.

Definition. A knot mosaic is *space-efficient* if it is reduced and if the tile number has been minimized through a sequence of planar isotopy moves.

Definition. A knot mosaic is *minimally space-efficient* if it is minimal and space-efficient.

Diagram space-efficiency is more rigid since we cannot change the location of crossings and is dependent upon the locations of the crossings within the knot mo-

saic diagram. Space-efficiency minimizes the tile number within the knot mosaic, moving, removing, or adding crossing tiles if necessary. To be minimally space-efficient, a knot mosaic must realize both the mosaic number and the minimal mosaic tile number of the knot or link. That is, if we have a knot K with mosaic number n , a knot mosaic of K is minimally space efficient if it is an n -mosaic and uses the smallest tile number possible on an n -mosaic. A knot mosaic of K could be space-efficient on any k -mosaic, with $k \geq n$.

The knot mosaic depicted in Figure 3(a) is not diagram space-efficient or space-efficient because we can take the arc that passes through the bottom, right corner tile of the mosaic and push it into the diagonally adjacent tile location, thus decreasing the number of non-blank tiles used in the mosaic, and the result is the knot mosaic depicted in Figure 3(d), which is both diagram space-efficient and minimally space-efficient. The knot mosaic in Figure 3(b) is diagram space-efficient but not space-efficient.

There are several things we can see immediately about these various versions of space-efficiency. As the examples in Figure 3 illustrate, a knot mosaic that is diagram space-efficient need not be reduced or minimal. A knot mosaic that is space-efficient must be reduced but need not be minimal. A knot mosaic that is minimally space-efficient must be reduced and minimal. On a minimally space-efficient knot mosaic, the minimal mosaic tile number of the depicted knot must be realized, but the tile number of the knot might not be realized. There may be a larger, non-minimal knot mosaic that uses fewer non-blank tiles. We provide an example of this later in the proof of Theorem 26. As for the relationship between these types of space-efficiency, we see that a diagram space-efficient knot mosaic need not be space-efficient, and a space-efficient knot mosaic need not be minimally space-efficient. However, the converse relationships follows directly from the definitions.

Proposition 1. *If a knot mosaic is minimally space-efficient, then it is diagram space-efficient and space-efficient.*

Proposition 2. *If a knot mosaic is space-efficient, then it is diagram space-efficient.*

Proposition 3. *If a knot mosaic is minimally space-efficient, then both the mosaic number and minimal mosaic tile number are realized in the mosaic.*

Our goal in this paper is to find minimally space-efficient knot mosaic diagrams for prime knots. Because of this and the previous proposition, our primary focus throughout the this paper will be on the minimal mosaic tile number of the knot or link, not the tile number of the knot. However, as we will see, there are many knots and links for which the tile number and minimal mosaic tile number are the same.

3 Useful Conventions, Terminology, and Counting Tools

As we seek to determine the tile numbers of knots and find minimally space-efficient knot mosaics for them, we will be working with a large number of possible placements

of tiles on a mosaic. To help us simplify explanations and figures, we adopt a few conventions. In particular, we will make use of *nondeterministic tiles* when there are multiple options for the tiles that can be placed in specific tile locations of a mosaic. We will usually denote these as dashed arcs or line segments on the tile. Some examples of these are shown in the first five tiles of Figure 4.

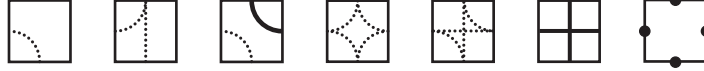


Figure 4: Examples of nondeterministic tiles.

The first tile in Figure 4 could be a single arc tile or a blank tile. The second one could be a single arc tile or a line segment tile. The third tile could be a single arc tile or a double arc tile, but the depicted solid arc is necessary. The fourth one could be any single arc or double arc tile. The fifth one could be a double arc tile or a crossing tile. The sixth tile shown in Figure 4 must be a crossing tile, but the crossing type is not yet determined. A point on the edge of a tile indicates a required connection point for the tile. The last tile in Figure 4 must have four connection points, and therefore, that tile must be either a double arc tile or a crossing tile.

If there is a connection point at the top or bottom of a tile, we may say that there is a connection point *entering* the row that contains that tile. Similarly, a connection point is entering a column if there is a connection point at the right or left of a tile in that column. The connection point may be referred to as an *entry point* for the row or column. For example, if there is a connection point between a tile in the third row and a tile in the fourth row of a mosaic, then that connection point is an entry point for the third row and an entry point for the fourth row. If a row or column of a mosaic has at least one non-blank tile in it, we may say that the row or column is *occupied*.

The *inner board* of an $n \times n$ mosaic is the $(n - 2) \times (n - 2)$ array of tiles that remain after removing the outermost rows and columns. The tiles in the outer most rows and columns are referred to as *boundary tiles*. The first and last boundary tiles in the first and last row of the mosaic are called *corner tiles*. Suppose there are two adjacent single arc tiles that share a connection point, and the other connection points enter the same adjacent row or column. The four options are shown in Figure 5, and we will refer to these collectively as *caps* and individually as *top caps*, *right caps*, *bottom caps*, and *left caps*, respectively.



Figure 5: A top cap, right cap, bottom cap, and left cap, respectively.

Equipped with this terminology, we consider the following lemmas that will

assist us in counting the minimum number of non-blank tiles necessary to create knot mosaics. We point out that some of these apply to mosaics of any knots and links, while others only apply to mosaics of prime knots. This first lemma tells us that we can create all of our knot mosaics without using the corner tile locations. Because of this, we will assume that the corners of any space-efficient mosaic are blank tiles. Because the outer rows and columns need not be occupied, we point out that this result actually applies to the first (and last) occupied row and column. In other words, we may assume that the first tile and the last tile in the first occupied row and column is a blank tile, and similarly for the last occupied row and column.

Lemma 4. *Suppose we have a reduced, diagram space-efficient n -mosaic with $n \geq 4$ and no unknotted, unlinked link components. Then the four corner tiles are blank T_0 tiles or can be made blank via a planar isotopy move. The same result holds for the first and last tile location of the first and last occupied row and column.*

Proof. We prove that the top, left corner must be blank, and the proof for the remaining three corners of the mosaic is similar. In addition to the top, left corner, also consider the tile that is diagonally adjacent to it, which is the top, left corner of the inner board of the mosaic. Our focus will be on the upper, left 2×2 corner sub-array that contains both of these tiles. We simply run through the eleven possible mosaic tiles that could be placed in the inner board corner. If the corner of the inner board is a blank tile, either all four tiles in the sub-array are blank or they are as depicted in the first case of Figure 6. The arc tile in the outer corner can be pushed into the inner board corner, leaving the outer corner blank without changing the tile number. In all of the other cases, either the outer corner is blank, or we can show that the mosaic has a trivial unlinked component, is not reduced, or is not diagram space-efficient. That is, there is an unlinked component, or we can use a series of planar isotopy moves to remove an unnecessary loop or decrease the tile number of the mosaic. Each case, where the outer corner is not blank and the corner of the inner board is one of the eleven possible mosaic tiles T_0, \dots, T_{10} , respectively, is depicted in Figure 6. In each of these cases, except the first one, the tile number is decreased or there is a trivial unlinked component.

We note that none of this argument hinges on the fact that the assumed blank tile must be in the first row and first column. It only requires that the rows above it and the columns to the left of it are blank. Thus, the result applies not only to the tile in the first row and first column, but also to the tile in the first occupied row and column. \square

Lemma 5. *For any knot mosaic, if a row (or column) is occupied, then there are at least two non-blank tiles in that row (or column). In fact, there are an even number of entry points between any two rows (or columns).*

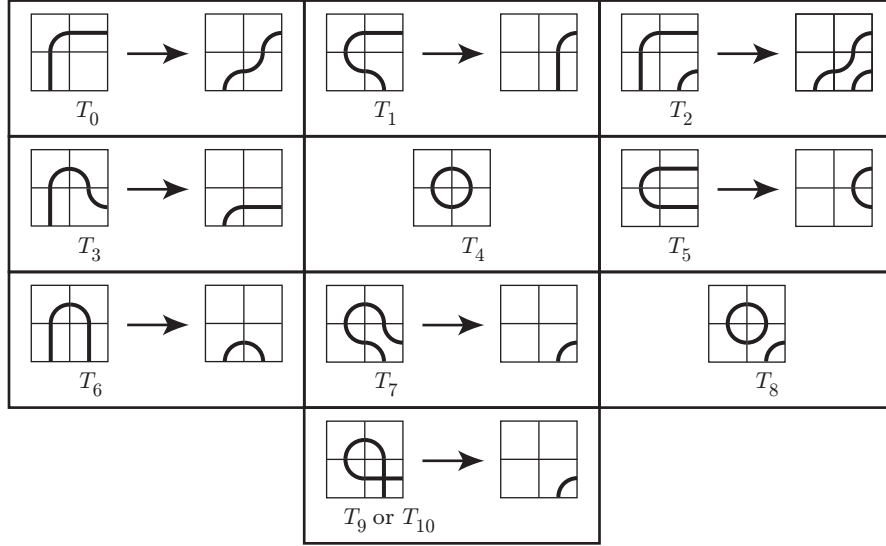


Figure 6: All possible upper, left 2×2 sub-arrays if the upper, left corner is not blank.

Proof. This lemma should be quite obvious, as knots and link components are simple closed curves. If there is an entry point from Row A into Row B, then a strand of the knot or a link component has entered Row B. In order to connect back to the rest of the knot or link component and complete the circle, that strand must pass back into Row A at some other entry point, necessarily on some other tile, and these entry points must come in pairs. The same is true for columns. \square

Lemma 6. *Suppose we have a reduced, diagram space-efficient n -mosaic with $n \geq 4$ and no unknotted, unlinked link components. If there is a cap in any row (or column), then the two tiles that share connection points with the cap must have four connection points. The same result holds if the arc tiles in the cap are not adjacent but have one or more line segment tiles between them.*



Figure 7: If there is a single strand of a knot or link in a row (or column) with both entry points coming from the same row (or column), then the tiles that share these entry points must have four connection points.

Proof. This proof will focus on rows, and the result for columns follows via a rotation of the mosaic. Suppose we have a top cap, as in the first diagram of Figure

7. We need to prove that the two tiles just below it must both have four connection points. Assume at least one of these tiles only has two connection points. Then each of the possibilities, except those resulting in a trivial unlinked link component, are shown in Figure 8, and they are either not diagram space-efficient, as the tile number can be decreased, or have an unnecessary loop. So both tiles connecting to the arcs must have four connection points. The cases involving the other caps are covered by a rotation of this.

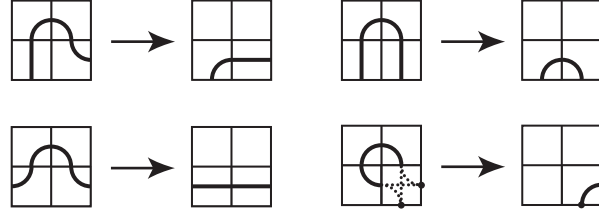


Figure 8: If tiles in the second row do not both have four connection points, the mosaics are not space-efficient or have an unnecessary loop.

Now suppose the two arcs in the cap are not adjacent but connected by a horizontal line segment, as in the second diagram in Figure 7. Again we need to prove that the two tiles below the arc tiles must both have four connection points. Assume at least one of these tiles only has two connection points. Each of the possibilities, except those resulting in a trivial unlinked link component, are shown in Figure 9, and none of them are diagram space-efficient. So both tiles connecting to the arcs must have four connection points. The cases are similar if there is more than one

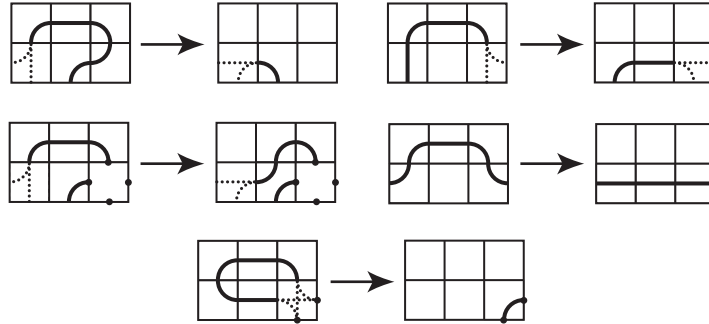


Figure 9: If tiles in the second row do not both have four connection points, they are not space-efficient.

line segment tile connecting the two arc tiles, and again, the cases involving the other caps are covered by a rotation of this. \square

This next lemma tells us that, in a space-efficient mosaic of a prime knot, we may assume that any occupied row or column with less than four non-blank tiles has exactly two non-blank tiles.

Lemma 7. *Suppose we have a space-efficient mosaic of a prime knot. If there is an occupied row (or column) with less than four non-blank tiles, then the mosaic can be simplified so that the row (or column) has exactly two non-blank tiles in the form of a cap.*

Proof. As before, this proof will focus on rows, and the result for columns follows via a rotation of the mosaic. Suppose we have a space-efficient mosaic of a prime knot. If a row is occupied and there are less than four non-blank tiles in the row, then there are either two or three non-blank tiles in the row by Lemma 5. By the same lemma, there must be an even number of entry points at the top of the row and at the bottom of the row. As there are no more than three non-blank tiles, this means there are either zero or two entry points at the top of the row and zero or two entry points at the bottom of the row. If all of the non-blank tiles are vertical segment tiles, this row can be collapsed by shifting the rows below it upward. All other possibilities, up to rotation or reflection, are shown in Figure 10.



Figure 10: Only possibilities to have less than four non-blank tiles in a single row.

The last possibility results in at least two (unlinked) link components. The next to last possibility is not reduced, as the crossing can be removed by a flip. Because the mosaic must depict a prime knot, each of the third, fourth, and fifth possibilities in Figure 10 is not space-efficient, as the portion of the knot either above or below this row must be unknotted and would simplify to one of the first two possibilities.

Consider one of those first two possibilities. Because there are no other non-blank tiles in this row and the knot mosaic does not depict a link, we know all tiles above this row must be blank. If the row under consideration looks like the second option in Figure 10, with a horizontal line segment between the two single arc tiles, then we claim that the mosaic is either not space-efficient or the horizontal segment can be collapsed in a way that does not change the tile number. To show this, we use Lemma 6, which tells us about the row below this one. These two rows must be as in the second picture of Figure 7, with two horizontal segment tiles in the same column. The tiles above the horizontal segments in this column are blank. If all of the tiles in this column below the horizontal segments are blank or horizontal line segment tiles, then the mosaic is not space-efficient as the tile number can be decreased by collapsing this column and shifting all of the right-most columns to the left by one column. So this is not an option. Now consider the first tile in this column that is not blank or a horizontal segment. Because the tile above it is blank or a horizontal line segment, this tile can only be a single arc tile T_1 or T_2 . In either case, the horizontal segment tiles can be collapsed without changing the tile number, or the mosaic is not space-efficient, as the tile number can be decreased

by collapsing the horizontal segments. If the first non-blank, non-horizontal tile is the T_1 arc tile, then the collapse of the horizontal segments is done as in one of the options in Figure 11, possibly with more blank or horizontal segment tiles above the arc tile. Rotations and reflections of these cover all other cases.

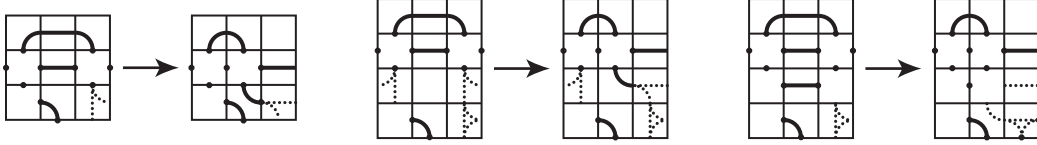


Figure 11: Collapsibility of horizontal segments.

Therefore, since the mosaic is space-efficient, we can always alter the given mosaic via planar isotopy moves so that any row or column with less than four non-blank tiles has exactly two non-blank tiles in the form of a cap, and this alteration does not change the tile number. \square

Because of this lemma, in a space-efficient mosaic of a prime knot, we may assume that every occupied row and column has at least four non-blank tiles or can be simplified to a single cap. Note that we are not saying that a space-efficient mosaic of a prime knot cannot have three non-blank tiles in a row or column. However, if there is such a row or column, we can reduce the number of non-blank tiles in that row or column to two via a planar isotopy move that does not change the overall tile number of the mosaic.

Corollary 8. *Every space-efficient mosaic of a prime knot can be drawn so that every row and column has either 0, 2, 4, or more non-blank tiles.*

Lemma 9. *Suppose we have a reduced, diagram space-efficient n -mosaic of a knot or link. Then the first occupied row of the mosaic can be simplified so that it is made up of top caps only. In fact, there will be k top caps for some k such that $1 \leq k \leq (n - 2)/2$. Similarly, the last occupied row is made up of bottom caps, and the first and last occupied columns are made up of left caps and right caps, respectively.*

Proof. Because we are considering the first occupied row of the mosaic, there can be no connection points along the top of the row. So the row must consist entirely of top caps or T_1 and T_2 single arc tiles separated by any number of horizontal segment tiles. If there is only one horizontal segment tile between the arc tiles, this can be reduced to a top cap without changing the tile number via the same argument in the proof of Lemma 7. If there are two horizontal segment tiles between the arc tiles, then we can eliminate them via a planar isotopy move without changing the tile number, as depicted in Figure 12.

Consider the two columns that contain the horizontal segment tiles in the first row. If every tile position in either of these two columns is filled with horizontal

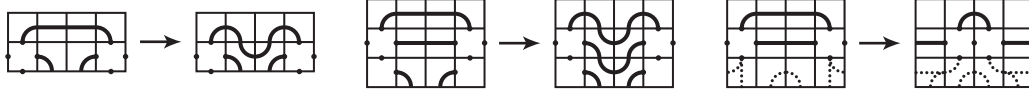


Figure 12: Simplifying to top caps.

segment tiles or blank tiles, then the mosaic is not space-efficient, as we could collapse the column(s). So we know that eventually there will be single arc tiles below the two horizontal segments. We have shown in Figure 12 the planar isotopy moves for the cases where this occurs in the second or third occupied row. If it happens in a later row, the moves are similar.

If there are more than two horizontal segment tiles between the arc tiles in the first occupied row, we can eliminate consecutive pairs as above, reducing the number of horizontal segment tiles between two arc tiles to one or none, and we can eliminate the single horizontal line segments as we did in the proof of Theorem 7. In any case, we are able to reduce everything to a collection of top caps.

Because there are n tile locations in the first occupied row and, by Lemma 4, we can assume the first and last tiles in this row are blank, there are only $n - 2$ tiles to place the top caps. Therefore, there are at most $(n - 2)/2$ top caps. Rotations of this prove the result for the first and last occupied columns and rows. \square

Lemma 10. *Suppose we have a space-efficient mosaic of a prime knot with at least five occupied rows. Then every occupied row except the first two and last two occupied rows has at least five non-blank tiles.*

Proof. We begin with a space-efficient mosaic of a prime knot with at least five occupied rows. By Lemma 9, we know that the first occupied row is made up of top caps.

If there are two (or more) top caps in the first occupied row, then the row below this will have four (or more) tiles with four connection points, a T_1 arc tile, a T_2 arc tile, and possibly more non-blank tiles. This gives at least six entry points into the third occupied row, forcing at least six non-blank tiles in the third occupied row. If there is only one top cap in the first occupied row, we know from Lemma 6 that there are at least four non-blank tiles in the second occupied row. Exactly two of them have four connection points and all of the other non-blank tiles are either horizontal segment tiles or single arc tiles. However, one of them must be the single arc tile T_1 , and one must be the single arc tile T_2 . This gives us four entry points and four non-blank tiles in the third occupied row.

In fact, to avoid space-inefficiency, composite knots, and multi-component links, we know there must be at least four connection points between any two rows, except possibly between the first two occupied rows and between the last two occupied rows. This means that in any one of these intermediate rows, there must be at least four connection points along the top of the row and at least four connection points along

the bottom of the row. If there are more than four in any given row, then there are more than four non-blank tiles in that row. Suppose there are exactly four connection points along the top and along the bottom of one of these intermediate rows. If the four connection points at the top of this row are vertically aligned with the four connection points at the bottom of the row, then these four non-blank tiles must all be vertical segment tiles, and the resulting mosaics would not be space-efficient. Thus, they are not vertically aligned, and there are at least five non-blank tiles in this row. Therefore, other than the first two occupied rows and the last two occupied rows, every row must have at least five non-blank tiles. \square

Several of these lemmas combine to provide a bound for the tile number. We have an upper bound for the tile number of a general n -mosaic of any knot or link, and we have a lower bound for an n -mosaic of any prime knot.

Theorem 11. *For $n \geq 4$, suppose we have a space-efficient n -mosaic of a knot or link K with no unknotted, unlinked link components, and either every row or every column of the mosaic is occupied. If n is even, then the tile number of the mosaic is less than or equal to $n^2 - 4$. If n is odd, then the tile number of the mosaic is less than or equal to $n^2 - 8$. If K is a prime knot, then the tile number is greater than $5n - 8$.*

Proof. Suppose we have a space-efficient n -mosaic of K in which either every row or every column is occupied. By Lemma 4, we know we do not need to use the corners of the mosaic. In the case where n is odd, Lemma 5 forces one more blank tile in each outer row and column because we can only have an even number of non-blank tiles in each of these. Therefore, the tile number of this mosaic must be less than or equal to either $n^2 - 4$ or $n^2 - 8$, depending on whether n is even or odd.

Now suppose that K is a prime knot, and assume every row of the mosaic is occupied. By Lemma 7, we may assume that the first row of the mosaic either has at least four non-blank tiles or has exactly two non-blank tiles in it, a top cap. Assuming the latter, Lemma 6 tells us that the next row down has at least four non-blank tiles. Lemma 10 tells us that the rest of the rows must have at least five non-blank tiles, except possibly the last and next to last rows. At a minimum, since all rows are occupied, the last row must have at least two non-blank tiles (a bottom cap), and the next to last row has at least four non-blank tiles. Thus there are at least two non-blank tiles in the first and last rows, at least four non-blank tiles in the second and next to last rows, and at least five non-blank tiles in each of the $n - 4$ intermediate rows, providing a minimum of $5n - 8$ non-blank tiles in the n -mosaic. A rotation of this gives the same result if every column is occupied. \square

Corollary 12. *Suppose we have a knot or link K with mosaic number $m(K) = n$ for $n \geq 4$ and no unknotted, unlinked link components. If n is even, then $t(K) \leq n^2 - 4$. If n is odd, then $t(K) \leq n^2 - 8$. If K is a prime knot, then $t(K) \geq 5n - 8$.*

Proof. Let K be a knot or link with mosaic number n . Then we know K can be drawn on an n -mosaic (or larger) but not a smaller mosaic, and the bounds for the $t(K)$ follow immediately from the theorem. Although the tile number of a prime knot may or may not occur on a minimal mosaic, larger mosaics have larger lower bounds on the tile number of the mosaic. So the tile number of any prime knot K with mosaic number n will never be smaller than $5n - 8$. \square

4 Tile Numbers of Small Knot Mosaics

Let us first consider the smallest mosaics, that is, n -mosaics with $n \leq 5$. We begin with the unknot, which has mosaic number 2.

Theorem 13. *The tile number (and minimal mosaic tile number) of the unknot is $t(\text{unknot}) = 4$.*

Proof. The least number of non-blank tiles necessary to create the unknot is four, and this is shown on a minimal mosaic in the first mosaic of Figure 13. If there are less than four tiles, the mosaic would not be suitably connected. \square

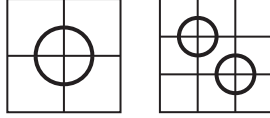


Figure 13: The unknot and two component unknotted unlink.

Kauffman and Lomonaco [4] show that the only knots or links that fit on a 3-mosaic are the unknot or the two component unknotted unlink. The latter of which has tile number and minimal mosaic tile number 7, as seen in Figure 13. All other knots and links have mosaic number 4 or more.

Theorem 14. *The tile number (and minimal mosaic tile number) of any knot or nontrivial link with mosaic number 4 is 12.*

Proof. For prime knots, this is a direct result of Corollary 12, which says that when the mosaic number is 4, the tile number is bounded above and below by 12. For composite knots and links with mosaic number 4, Corollary 12 only says that the upper bound is 12. As long as the link is nontrivial, there must be at least two crossing tiles in the mosaic. For space-efficiency, the mosaic cannot have any unnecessary loops that can be removed via a Reidemeister Type I move. Then any suitably connected knot mosaic with at least two crossing tiles must have tile number at least 12, and up to symmetry, the only options are shown in Figure 14. \square

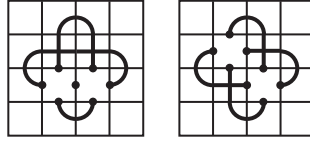


Figure 14: A knot mosaic with at least two crossings has tile number at least 12.

Corollary 15. *The tile number (and minimal mosaic tile number) of the trefoil knot is $t(3_1) = 12$.*

We now begin our exploration of 5-mosaics. In particular, we seek to find the possible tile numbers of space-efficient 5-mosaics, and find the tile number of all knots and links with mosaic number 5.

For prime knots, this is simple. For a prime knot with mosaic number 5, Corollary 12 tells us that the tile number is bounded above and below by 17. For a composite knot or link K with mosaic number 5, Corollary 12 provides an upper bound $t(K) \leq 17$. Just a little more work is required to show that this is also the lower bound.

Theorem 16. *The tile number (and minimal mosaic tile number) of any knot or link K with mosaic number 5 and no unknotted, unlinked components is $t(K) = 17$. This includes the prime knots 4_1 , 5_1 , 5_2 , 6_1 , 6_2 , and 7_4 . Moreover, any space-efficient 5-mosaics of K has a layout as shown in Figure 15.*

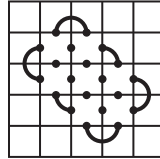


Figure 15: Only possible layout for a space-efficient 5-mosaic.

Proof. Because the mosaic number of K is 5, either every row or every column must be occupied. Assume every row of the mosaic is occupied. By Lemma 9, we may assume that the first row of the 5-mosaic has two non-blank tiles, a top cap. By Lemma 6, the second row must have at least four non-blank tiles. Similarly, the last row has two non-blank tiles, and the next to last row has at least four non-blank tiles. Now we observe the middle row. There are at least four entry points at the top of this row and four entry points at the bottom of it. If there are exactly four non-blank tiles in this row, then this means that the entry points at the top of the row are vertically aligned with the entry points at the bottom of the row, and the four non-blank tiles in this row must be vertical line segments, which means that

the mosaic is not space-efficient. Therefore, there must be five non-blank tiles in the middle row, giving us a minimum tile number of 17. \square

The minimally space-efficient mosaics of the prime knots K with mosaic number $m(K) = 5$ and tile number $t(K) = 17$ are provided in the table of mosaics in Section 8.

5 Tile Numbers of Knots with Mosaic Number 6

Now we wish to do for 6-mosaics what we have done for the smaller mosaics. That is, we wish to find the possible tile numbers of all 6-mosaics. However, because some of the lemmas in Section 3 are only known to apply to mosaics of prime knots, we will restrict ourselves to only looking at 6-mosaics of prime knots. Theorem 11 gives us the bounds for the tile number of any space-efficient 6-mosaic or 7-mosaic depicting a prime knot. In particular, suppose we have a prime knot K on a space-efficient n -mosaic. If $n = 6$, then the tile number t of the mosaic is $22 \leq t \leq 32$. If $n = 7$, then the tile number t of the mosaic is $27 \leq t(K) \leq 41$. This leads us to some immediate corollaries to Theorem 11.

Corollary 17. *For any prime knot K with mosaic number $m(K) \leq 6$, if the minimal mosaic tile number $t_m(K) \leq 27$, then the tile number of K equals the minimal mosaic tile number of K .*

Proof. We already knew this result for $m(K) \leq 5$. Since a 7-mosaic or larger cannot have tile number smaller than 27, we know that for any prime knot with mosaic number 6 and minimal mosaic tile number at most 27, the number of non-blank tiles cannot be decreased by placing it on a larger mosaic. \square

We can now determine the tile number of all prime knots with crossing number 7 or less and several prime knots with crossing number 8 or 9.

Corollary 18. *Let K be the 6_3 knot or any prime knot with crossing number $c(K) = 7$, except 7_4 . Then the tile number of K is $t(K) = 22$.*

Proof. We have given minimally space-efficient mosaics with tile number 22 for knot 6_3 and the seven crossing knots 7_1 , 7_2 , 7_3 , 7_5 , 7_6 , and 7_7 in the table of knots included in Section 8. Since the mosaic number of each of these knots is 6, we know that they cannot have a tile number smaller than 22. \square

One potentially interesting note is that, for all of the minimally space-efficient knot mosaics for each of these knots in Corollary 18, the crossing number was also realized except in one case. In order to obtain the minimally space-efficient knot mosaic for 7_3 , we had to use eight crossings. None of the possible minimally space-efficient knot mosaics with twenty-two non-blank tiles and exactly seven crossings

produced the knot 7_3 . The fewest number of non-blank tiles needed to represent 7_3 with only seven crossings is twenty-four, and one such mosaic is given in Figure 16, along with a minimally space-efficient mosaic of 7_3 with eight crossings. In summary, on a minimally space-efficient knot mosaic, for the tile number (or minimal mosaic tile number) to be realized, the crossing number need not be realized and, in some cases, cannot be realized.

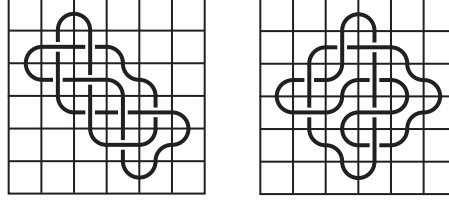


Figure 16: The 7_3 knot as a minimally space-efficient knot mosaic with eight crossing tiles and as a knot mosaic with seven crossing tiles.

Corollary 19. *Let K be one of the following knots with crossing number $c(K) = 8$ or $c(K) = 9$: $8_1, 8_2, 8_3, 8_4, 8_7, 8_8, 8_9, 8_{13}, 9_5$, or 9_{20} . Then the tile number of K is $t(K) = 22$.*

Proof. We have given minimally space-efficient mosaics with tile number 22 for each of these knots in the table of knots included in Section 8. Since the mosaic number of each of these knots is 6, we know that they cannot have a tile number smaller than 22. \square

Theorem 20. *If we have a space-efficient 6-mosaic of a prime knot K for which either every column or every row is occupied, then the only possible values for the tile number of the mosaic are 22, 24, 27, and 32. Furthermore, any such mosaic of K is equivalent (up to symmetry) to one of the mosaics in Figure 17.*

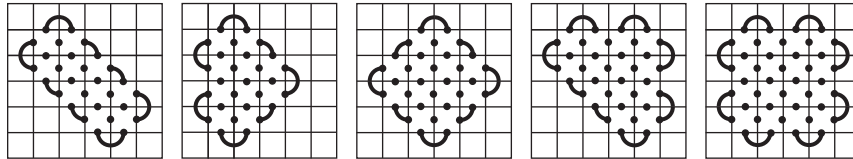


Figure 17: Only possible layouts for a space-efficient 6-mosaic.

Proof. If there are two non-blank tiles (one top cap) in the first row, we claim the second row must have four non-blank tiles. If it had more than four, there would be at least one horizontal segment tile in this row, and this will cause the mosaic to not be space-efficient. The same result holds for the second occupied row from the

bottom and the second occupied columns from the right or left. To prove the claim, we consider the possible locations of the top cap in the first occupied row.

Suppose there is a top cap in the first two tile positions after the corner tile. Then the first tile in the second occupied row must be a single arc tile T_2 , followed by two tiles with four connection points. If the next two tiles are both horizontal segment tiles, this forces the arc tile T_1 into the last position in this row, which is necessary part of a right cap, and the previous tile position with the horizontal segment should have had four connection points by Lemma 6. If there is only one horizontal segment, then the fifth tile position is the arc tile T_1 . Assume this is not part of a cap, and look at the tile directly below the horizontal segment. Because there is no connection point at the top of this tile, it can only be a horizontal segment, T_1 , T_2 , or blank tile. If it is a horizontal segment tile, then the mosaic is not space-efficient because either everything in this column is a horizontal segment or blank tile or the mosaic is as depicted in Figure 18(a), in which the entire upper, left 3×3 corner of the mosaic can be shifted to the right, collapsing the horizontal segments. If it is a T_1 tile, the knot is not space-efficient, as we can see in Figure 18(b), and the tile number can be decreased by pushing the horizontal segment and T_1 tiles from the second row into the T_1 tile in the third row. If it is a T_2 tile or blank tile, the mosaic must be as in Figure 18(c), and either the knot is not prime or the mosaic is not space-efficient, as shown by the dashed line cutting through the knot.

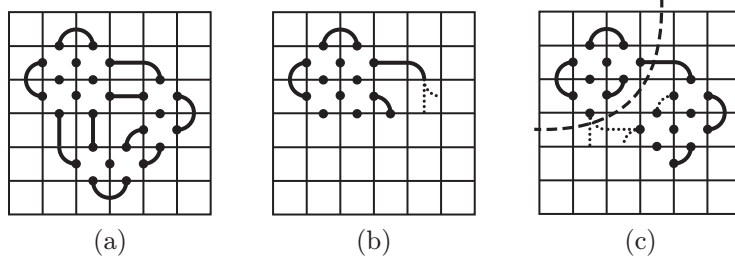


Figure 18: Possible configurations with a horizontal segment in the fourth tile position of the second row.

If there is a top cap in the second and third tile positions after the corner tile, it is easy to see that a horizontal segment tile is not allowed in the second row. If there was one, this would force a single arc tile into a boundary column, which is necessary part of a cap, and the tile position with the horizontal segment should have had four connection points by Lemma 6.

Suppose every row of the mosaic is occupied. By Lemma 9, the first row has either one or two top caps, that is, two or four non-blank tiles. By Lemma 6, we know that the tiles directly below the top caps must have four connection points. There must also be at least one single arc tile T_1 and one single arc tile T_2 . Thus, if there are four non-blank tiles in the first row, the second row must have six non-

blank tiles. We just showed that if there are two non-blank tiles in the first row, the second row has exactly four non-blank tiles. By Lemma 10, we know that the middle two rows have five or six non-blank tiles. The non-blank tiles in the last two rows are counted as they were in the first two rows. Analogously, we know how many non-blank tiles can be in each of the columns. If not every row of the mosaic is occupied, then every column must be, and a similar argument applies. With all of this in mind, the five layouts depicted in the theorem are the only possible configurations, up to rotation, reflection, or translation, of the non-blank tiles.

Now we turn our attention to the connection points. Notice that all of the nondeterministic tiles must have four connection points. Most of them are there because Lemma 6 requires it. In the second layout, for example, all of the connection points are required by Lemma 6. In the remaining four layouts, the only connection points that are not required by Lemma 6 are the four connection points on the tile edges that meet at the center point of the mosaic. In the first, fourth, and fifth layouts, if any of these four connection points are missing, then either the knot is not prime or the mosaic is not space-efficient because there would be only two connection points between the third and fourth columns. In the third layout, if any of these four connection points is missing, then there would only be eight tile locations with four connection points. If all eight of these are crossing tiles, the result is a two component link. If less than eight of them are crossings, we know the mosaic is not space-efficient because every prime knot with seven crossings or less has tile number less than 24. \square

The following theorems form a summary of the final results of this paper and give the tile number or minimal mosaic tile numbers of every prime knot with mosaic number 6. We provide the proof of these theorems in Section 6. For now, we see what the results are. We provide minimally space-efficient knot mosaics for every prime knot with mosaic number less than or equal to 6 in the table of knots in Section 8. We have already listed several prime knots with tile number 22. This next theorem asserts that the list is complete.

Theorem 21. *The only prime knots K with tile number $t(K) = 22$ are:*

- (a) 6_3 ,
- (b) $7_1, 7_2, 7_3, 7_5, 7_6, 7_7$,
- (c) $8_1, 8_2, 8_3, 8_4, 8_7, 8_8, 8_9, 8_{13}$,
- (d) 9_5 , and 9_{20} .

Theorem 22. *The only prime knots K with tile number $t(K) = 24$ are:*

- (a) $8_5, 8_6, 8_{10}, 8_{11}, 8_{12}, 8_{14}, 8_{16}, 8_{17}, 8_{18}, 8_{19}, 8_{20}, 8_{21}$,
- (b) $9_8, 9_{11}, 9_{12}, 9_{14}, 9_{17}, 9_{19}, 9_{21}, 9_{23}, 9_{26}, 9_{27}, 9_{31}$,

- (c) $10_{41}, 10_{44}, 10_{85}, 10_{100}, 10_{116}, 10_{124}, 10_{125}, 10_{126}, 10_{127}, 10_{141}, 10_{143}, 10_{148}, 10_{155}$ and 10_{159} .

Again we note that the minimally space-efficient mosaics for $8_1, 8_3, 8_6, 8_7, 8_8$, and 8_9 must have nine crossing tiles. None of the possible minimally space-efficient knot mosaics with exactly eight crossings produce these knots. Similarly, the minimally space-efficient mosaics for $9_{12}, 9_{19}, 9_{21}$, and 9_{26} require ten crossings.

Theorem 23. *The only prime knots K with mosaic number $m(K) = 6$ and tile number $t(K) = 27$ are:*

- (a) 8_{15}
- (b) $9_1, 9_2, 9_3, 9_4, 9_7, 9_9, 9_{13}, 9_{24}, 9_{28}, 9_{37}, 9_{46}, 9_{48}$,
- (c) $10_1, 10_2, 10_3, 10_4, 10_{12}, 10_{22}, 10_{28}, 10_{34}, 10_{63}, 10_{65}, 10_{66}, 10_{75}, 10_{78}, 10_{140}, 10_{142}, 10_{144}$,
- (d) $11a_{107}, 11a_{140}$, and $11a_{343}$.

Notice that the previous theorem is only referring to prime knots with mosaic number 6. There are certainly prime knots with tile number 27 and mosaic number 7 that are not included in this theorem. Also notice that, up to this point, we have determined the tile number for every prime knot with crossing number 8 or less. For knots with crossing number 11 or higher, we use the Dowker-Thistlethwaite name of the knot.

Again we claim that the minimally space-efficient mosaics for $9_3, 9_4, 9_{13}, 9_{37}, 9_{46}$, and 9_{48} must have ten crossing tiles. The minimally space-efficient mosaics for $9_7, 9_9$, and 9_{24} must have eleven crossing tiles. None of the possible minimally space-efficient knot mosaics with exactly nine crossing tiles produce these knots. Similarly, the minimally space-efficient mosaics for $10_1, 10_3, 10_{12}, 10_{22}, 10_{34}, 10_{63}, 10_{65}, 10_{78}, 10_{140}, 10_{142}$, and 10_{144} require eleven crossing tiles.

Theorem 24. *The only prime knots K with mosaic number $m(K) = 6$ and minimal mosaic tile number $t_m(K) = 32$ are:*

- (a) $9_{10}, 9_{16}, 9_{35}$,
- (b) $10_{11}, 10_{20}, 10_{21}, 10_{61}, 10_{62}, 10_{64}, 10_{74}, 10_{76}, 10_{77}, 10_{139}$,
- (c) $11a_{43}, 11a_{44}, 11a_{46}, 11a_{47}, 11a_{58}, 11a_{59}, 11a_{106}, 11a_{139}, 11a_{165}, 11a_{166}, 11a_{179}, 11a_{181}, 11a_{246}, 11a_{247}, 11a_{339}, 11a_{340}, 11a_{341}, 11a_{342}, 11a_{364}, 11a_{367}$,
- (d) $11n_{71}, 11n_{72}, 11n_{73}, 11n_{74}, 11n_{75}, 11n_{76}, 11n_{77}, 11n_{78}$,
- (e) $12a_{119}, 12a_{165}, 12a_{169}, 12a_{373}, 12a_{376}, 12a_{379}, 12a_{380}, 12a_{444}, 12a_{503}, 12a_{722}, 12a_{803}, 12a_{1148}, 12a_{1149}, 12a_{1166}$,

- (f) $13a_{1230}$, $13a_{1236}$, $13a_{1461}$, $13a_{4573}$
(g) $13n_{2399}$, $13n_{2400}$, $13n_{2401}$, $13n_{2402}$, $13n_{2403}$.

Notice again our restriction to prime knots with mosaic number 6. Additionally, notice that this theorem only refers to the minimal mosaic tile number of the knot, not the tile number. Again, this is because we only know that these two numbers are equal when they are less than or equal to 27.

We claim that the minimally space-efficient mosaics for 9_{10} , 9_{16} , 10_{20} , 10_{21} , and 10_{77} need eleven crossing tiles. The minimally space-efficient mosaics for 9_{35} , 10_{11} , 10_{62} , 10_{64} , 10_{74} , 10_{139} , $11a_{106}$, $11a_{139}$, $11a_{166}$, $11a_{181}$, $11a_{341}$, $11a_{342}$, and $11a_{364}$ need twelve crossing tiles. And the minimally space-efficient mosaics for 10_{61} , 10_{76} , $11a_{44}$, $11a_{47}$, $11a_{58}$, $11n_{76}$, $11n_{77}$, $11n_{78}$, $11a_{165}$, $11a_{246}$, $11a_{339}$, $11a_{340}$, $12a_{119}$, $12a_{165}$, $12a_{169}$, $12a_{376}$, $12a_{379}$, $12a_{444}$, $12a_{803}$, $12a_{1148}$, and $12a_{1166}$ need thirteen crossing tiles.

These preceding theorems lead us to the following interesting consequences.

Corollary 25. *The prime knots with crossing number at least 9 not listed in Theorems 21, 22, 23, or 24 have mosaic number 7 or higher.*

Theorem 26. *The tile number of a knot is not necessarily equal to the minimal mosaic tile number of a knot.*

Proof. According to Theorem 24, the minimal mosaic tile number for 9_{10} is 32. However, on a 7-mosaic, this knot can be represented using only 27 non-blank tiles, as depicted in Figure 19. Also note that, as a 7-mosaic, this knot could be repre-

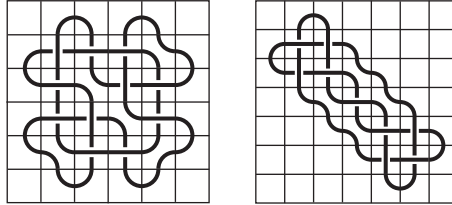


Figure 19: The 9_{10} knot represented as a minimally space-efficient 6-mosaic with minimal mosaic tile number 32 and as a space-efficient 7-mosaic with tile number 27.

sented with only nine crossings, whereas eleven crossings were required to represent it as a 6-mosaic. \square

6 Minimally Space-Efficient 6-Mosaics of Prime Knots

We know from Corollaries 18 and 19 that there are several prime knots known to have tile number 22. Now we want to show that this list is complete and, as a result,

which prime knots must have tile number greater than 22. By Theorem 20, we know all possible configurations of the tiles in a space-efficient 6-mosaic of a prime knot, given again in Figure 20.

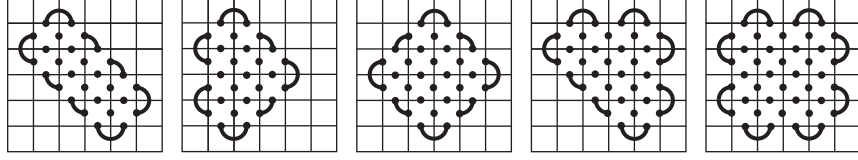


Figure 20: Possible layouts for a space-efficient 6-mosaic of a prime knot.

In order to determine the tile number or minimal mosaic tile number of prime knots with mosaic number 6, we need to determine which prime knots can be written as a knot mosaic with these layouts. We not only want to know which ones can be expressed, for example, on the layouts with twenty-two non-blank tiles, but we also want to know which ones cannot be expressed in this way. If a knot cannot be expressed on the layout with twenty-two non-blank tiles, then it must have tile number larger than 22. We will do this for each of the layouts in Figure 20. To help us with this, we make a few simple observations. All of these are easy to verify, and any rotation or reflection of these scenarios are also valid.

In order to easily refer to specific tile locations within a mosaic, on a 6×6 mosaic we label all of the boundary tiles except the corner tiles as $B^1 - B^{16}$, as depicted in Figure 21. We label the sixteen tiles on the inner board as $I^1 - I^{16}$, again shown in Figure 21.

	B^1	B^2	B^3	B^4	
B^{16}					B^5
B^{15}					B^6
B^{14}					B^7
B^{13}					B^8
	B^{12}	B^{11}	B^{10}	B^9	

	I^1	I^2	I^3	I^4	
	I^5	I^6	I^7	I^8	
	I^9	I^{10}	I^{11}	I^{12}	
	I^{13}	I^{14}	I^{15}	I^{16}	

Figure 21: Labels of the tiles in a 6×6 mosaic board.

Consider the upper, right 3×3 corner of any space-efficient mosaic of a prime knot with mosaic number 6 and tile number 22, 27, or 32. (That is, we are considering every option except those with tile number 24.) It must be one of the two options in Figure 22. All other 3×3 corners are a rotation of one of these. We will refer to the first option as a *partially filled block* and the second option as a *filled block*. We reiterate that each of the following observations refer to a space-efficient 6×6 mosaic of a prime knot.

Observation 1. For a partially filled block, the tile in position I^7 is either a crossing tile or double arc T_7 .

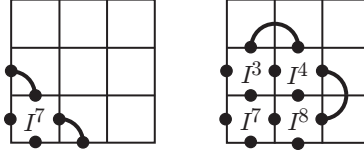


Figure 22: A partially filled block and a filled block, respectively.

This is easy to see, as it must be a tile with four connection points, and the only space-efficient mosaics that results from using the double arc T_8 are composite knots or links with more than one component. In Figure 23, the first two examples are valid possibilities, but the third one is not.

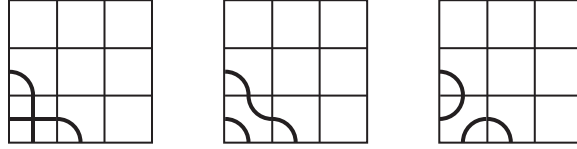


Figure 23: The first two examples are the only valid possibilities for a partially filled block, but the third one is not.

Observation 2. There must be at least two crossing tiles in a filled block.

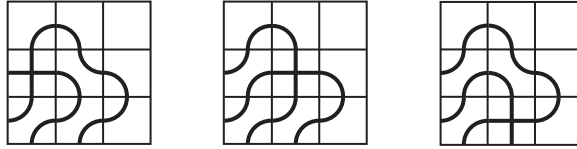


Figure 24: Diagram space-efficient filled blocks with one crossing in position I^3 , I^4 , or I^8 . None are space-efficient.

If there are no crossing tiles in positions I^3 , I^4 , I^7 , and I^8 of the mosaic, which means no crossings in the 3×3 corner of the mosaic, then the mosaic is not space-efficient or it is a link with more than one component. Each one that is not a link reduces to one of the last two partially filled block options in Figure 23. If there is only one crossing tile and it is in position I^3 , I^4 , or I^8 , then the mosaic is not space-efficient. If we complete all three of these options to be diagram space-efficient (as in Figure 24), they are equivalent to each other via a simple planar isotopy move that rolls the crossing through each of these positions, and they reduce to the first partially filled block in Figure 23. If there is only one crossing tile and it is in position I^7 , then the mosaic is also not space-efficient and reduces to either of the first two options in Figure 23.

Observation 3. For the filled block, there are only two distinct possibilities for two crossing tiles, two distinct possibilities for three crossing tiles, and one possibility for four crossing tiles, and they are shown in Figure 25.

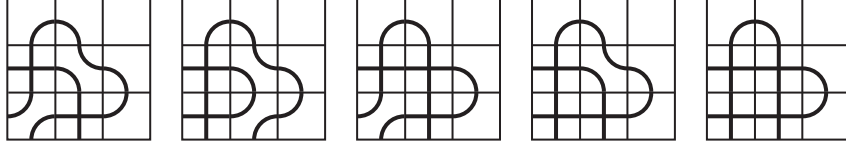


Figure 25: The only possible layouts for a filled block.

Observation 4. In any space-efficient 6-mosaic of a prime knot, there is at most one of a filled block with four crossing tiles or a filled block with two crossings in position I^3 and I^7 .

It is quite simple to verify that if there is more than one filled block with four crossings or more than one filled block with two crossings in positions I^3 and I^7 , the resulting mosaic must be a link with more than one component. If we use the indicated filled building block with two crossing tiles together with a filled block with four crossing tiles, the resulting mosaic will also be a link with more than one component. Several examples of these are pictured in Figure 26 with the second link component in each mosaic colored differently from the first link component.

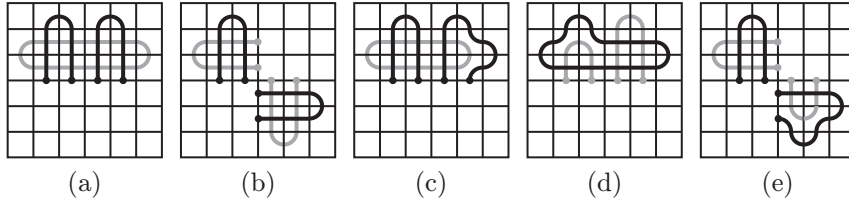


Figure 26: These layouts will always be multi-component links.

In the remainder of this section, we prove Theorems 21, 22, 23 and 24. In order to do this, we simply compile a list of the prime knots with crossing number at least 8 that can fit within each of the layouts given in Figure 20. (We already know the minimal mosaic tile number of every prime knot with crossing number 7 or less.) The process is simple, and the above observations help us tremendously. We will refer to the five filled blocks in Figure 25 together with the first two partially filled blocks in Figure 23 as *building blocks*. The observations provide a way for us to easily build all of the space-efficient 6-mosaics, unless the tile number is 24. That is, we can build all space-efficient 6-mosaics with tile number 22, 27, or 32 using the building blocks (or reflections and rotations of them). In the case of the mosaics with tile number 24, we look at all possible placements, up to symmetry, of eight

or more crossing tiles within the mosaics. We fill the remaining tile positions with double arc tiles so as to avoid composite knots and unnecessary loops. Regardless of the tile number, once the mosaics are completed, we then eliminate any links, any duplicate layouts that are equivalent to others via simple planar isotopy moves, and any mosaics for which the tile number can easily be reduced by a simple planar isotopy move.

We are now ready to prove Theorem 21 and find all of the prime knots with tile number 22.

Theorem 21. *The only prime knots K with tile number $t(K) = 22$ are:*

- (a) 6_3 ,
- (b) $7_1, 7_2, 7_3, 7_5, 7_6, 7_7$,
- (c) $8_1, 8_2, 8_3, 8_4, 8_7, 8_8, 8_9, 8_{13}$,
- (d) 9_5 , and 9_{20} .

Proof. We already know from Corollaries 18 and 19 that each of the knots listed have tile number 22. We also already know that every prime knot with crossing number less than or equal to seven has tile number less than or equal to 22. So our focus will be on prime knots with crossing number greater than seven. We simply build the tile configurations with twenty-two non-blank tiles in Figure 20 using the 3×3 building blocks, eliminate any that do not satisfy the observations, choose specific crossing types, and see what we get. Whatever prime knots with eight or more crossings are missing are the ones we know cannot have tile number 22. We begin with the first mosaic layout given in Figure 20 with twenty-two non-blank tiles and observe the possibilities that arise with eight crossings.

Up to symmetry, there are only six possible configurations of this layout with eight crossings, and they are given in Figure 27. Notice that some of these are links that can be eliminated, including Figures 27(d) and (f). Furthermore, Figures 27(b) and (c) are equivalent to each other via a simple planar isotopy move that shifts one of the crossing tiles to a diagonally adjacent tile position. This leaves us with only three possible distinct configurations of eight crossings from this first layout, Figures 27(a), (b), and (e).

Now we do the same thing with the second mosaic layout given in Figure 20 with twenty-two non-blank tiles. Up to symmetry, there are six possible configurations of this layout with eight crossings, and they are given in Figure 28. Again, a couple of these are links that we can eliminate, including Figures 28(d) and (f). Furthermore, Figures 28(b) and (c) are equivalent to each other via a simple planar isotopy move that shifts a single crossing to a diagonally adjacent tile. This leaves us again with only three possible configurations of eight crossings from this second layout, and they are again Figures 28(a), (b), and (e). Moreover, each one of these is equivalent

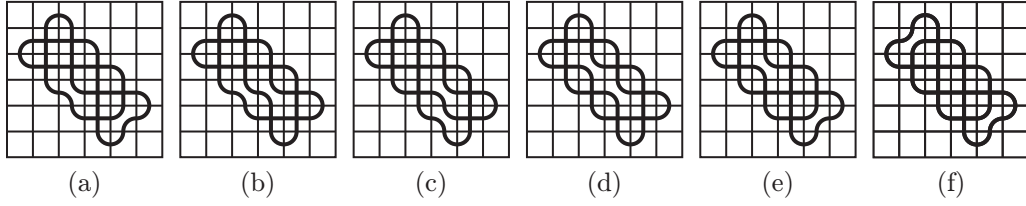


Figure 27: Possible placements of eight crossing tiles in the first layout with tile number 22.

to the corresponding mosaics in Figure 27 via a few simple planar isotopy moves that shift the crossings in the lower, left building block into the lower, right building block of the mosaic.

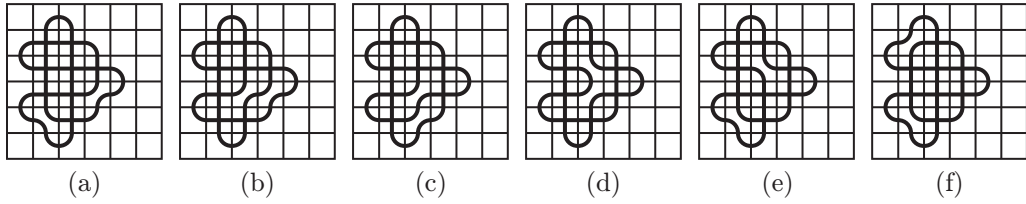


Figure 28: Possible placements of eight crossing tiles in the second layout with tile number 22.

This leaves us with only three distinct possible layouts for a minimally space-efficient 6×6 mosaic with eight crossings and tile number 22. We now need to choose the type of crossings for each crossing in these possibilities. If we choose crossings for the configuration in Figure 27(a) so that they are alternating, we get the 8_{13} knot. If we choose crossings for the configuration in Figure 27(b) so that they are alternating, we get the 8_4 knot. Finally, if we choose crossings for the configuration in Figure 27(e) so that they are alternating, we get the 8_2 knot. If we examine all possible non-alternating choices for each one, all of the resulting knots have seven crossings or less. (The minimally space-efficient knot mosaic for 7_3 must have eight crossing tiles and can be obtained by a choice of non-alternating crossings within any of the three distinct possible layouts in Figure 27.) The resulting knot mosaics are given in the table of knots in Section 8.

Now we go through the same process using nine crossing tiles using both of the mosaic layouts given in Figure 20 with twenty-two non-blank tiles. Up to symmetry, there are only four possible configurations of these layouts with nine crossings, and they are given in Figure 29. The mosaic in Figure 29(c) is equivalent to the mosaic in Figure 29(b) via a few simple planar isotopy moves that shifts the crossings in the lower, left building block into the lower, right building block of the mosaic. This leaves us with only three possible configurations of nine crossing tiles.

We now need to choose the type of crossings for each crossing in these possibil-

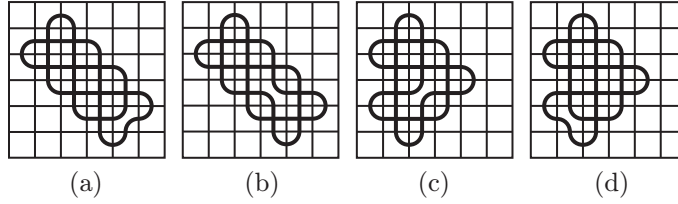


Figure 29: Possible placements of nine crossings with tile number 22.

ities. If we choose crossings for the configuration in Figure 29(a) so that they are alternating, we get the 9_{20} knot. If we examine all possible non-alternating choices for the crossings, most of the resulting knots have seven crossings or fewer, but we do get some additions to our list of prime knots with tile number 22 and crossing number 8. In particular, we get 8_7 , 8_8 , and 8_9 . (We also get 8_4 , which was previously obtained with only eight crossings.) If we choose crossings for the configuration in Figure 29(b) so that they are alternating, we get the 9_5 knot. Again, if we examine the possible non-alternating choices for the crossings, we get two additional prime knots with tile number 22 and crossing number 8, and they are 8_1 and 8_3 . Finally, if we choose crossings for the configuration in Figure 29(d), we get the exact same knots as we did for Figure 29(a). Mosaics for all of these resulting knots are given in the table of knots in Section 8.

Any mosaic with twenty-two non-blank tiles and ten crossing tiles is a link with more than one component. We have now found every possible prime knot with tile number 22 and eight or more crossings, and they are exactly those listed in Corollary 19. All other prime knots with crossing number at least eight must have tile number larger than 22. Therefore, these remaining prime knots must have tile number at least 24. \square

We now know precisely which prime knots have tile number 22 or less. Our next goal is to determine which prime knots have tile number 24 by providing the proof of Theorem 22.

Theorem 22. *The only prime knots K with tile number $t(K) = 24$ are:*

- (a) $8_5, 8_6, 8_{10}, 8_{11}, 8_{12}, 8_{14}, 8_{16}, 8_{17}, 8_{18}, 8_{19}, 8_{20}, 8_{21},$
- (b) $9_8, 9_{11}, 9_{12}, 9_{14}, 9_{17}, 9_{19}, 9_{21}, 9_{23}, 9_{26}, 9_{27}, 9_{31},$
- (c) $10_{41}, 10_{44}, 10_{85}, 10_{100}, 10_{116}, 10_{124}, 10_{125}, 10_{126}, 10_{127}, 10_{141}, 10_{143}, 10_{148},$
 10_{155} and $10_{159}.$

Proof. We search for all of the prime knots that have tile number 24. In this particular case, the observations at the beginning of this section do not apply, meaning we cannot use the building blocks as we did in the proof of Theorem 21. We know from Theorem 20 that any prime knot with tile number 24 has a space-efficient mosaic

like the third layout in Figure 20. Our goal is to compile a list of the prime knots with eight or more crossings that can fit within this layout of twenty-four non-blank tiles by looking at all possible placements of the crossings. Once this is completed, we will not only know which prime knots have tile number 24, but we will also know how many crossing tiles are necessary.

First, we look at all possible placements, up to symmetry, of eight crossings within the third mosaic shown in Figure 20. Second, we fill the remaining tile positions with double arc tiles so as to avoid composite knots and unnecessary loops. Third, we eliminate any links and any duplicate layouts that are equivalent to others via simple planar isotopy moves. In the end, we get seventeen possible layouts, which are shown in Figure 30. Not all of these will result in distinct knots, and in most cases it is not difficult to see that they will result in the same knot. However, we include all of them here because they differ by more than just simple symmetries or simple planar isotopy moves.

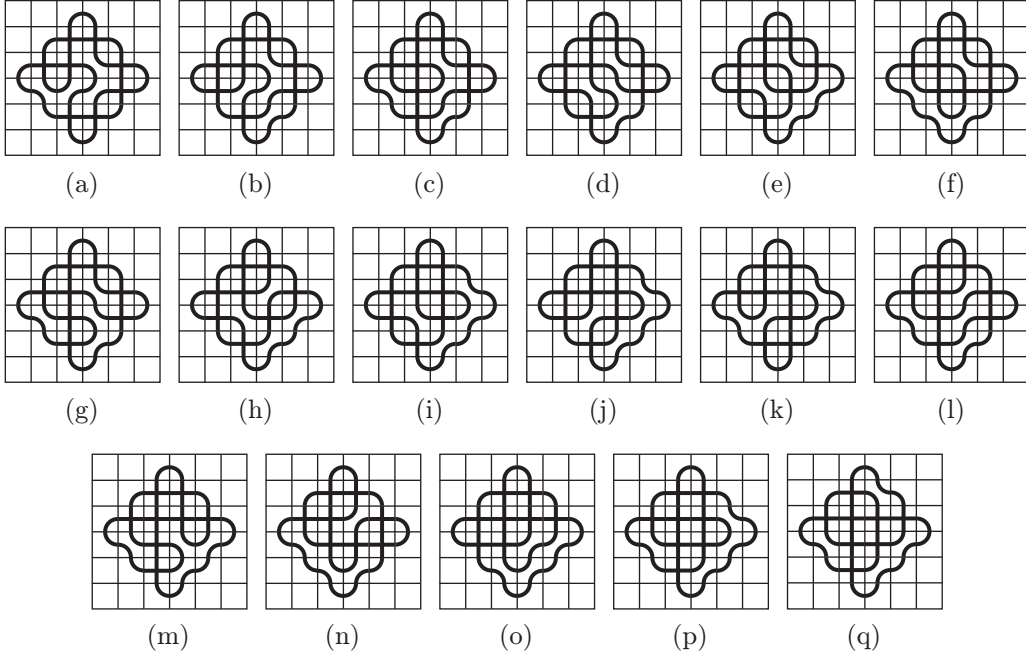


Figure 30: Only possible layouts, after elimination, with eight crossing tiles for a prime knot with tile number 24.

Choosing specific crossings so that the knots are alternating, we obtain only fourteen distinct knots as follows. Figure 30(a) is the 8_1 knot. Figures 30(b) and (c) are 8_2 . Figure 30(d) is 8_4 . Figure 30(e) is 8_5 . Figures 30(f) and (g) are 8_7 . Figure 30(h) is 8_8 . Figure 30(i) is 8_{10} . Figure 30(j) is 8_{11} . Figure 30(k) is 8_{12} . Figure 30(l) is 8_{13} . Figures 30(m) and (n) are 8_{14} . Figure 30(o) is 8_{16} . Figure 30(p) is 8_{17} . Figure 30(q) is 8_{18} . Not all of these have tile number 24. We already know 8_1 , 8_2 ,

8_4 , 8_7 , 8_8 , and 8_{13} have tile number 22. Each of the others have tile number 24. The non-alternating knots 8_{19} , 8_{20} , and 8_{21} are obtained from choosing non-alternating crossings in a few of these. Those pictured in the table of knots come from the layout in Figure 30(p). Mosaics for all of these resulting knots are given in the table of knots in Section 8. The only knots with crossing number 8 that we have not yet found are 8_6 and 8_{15} , and now we know that they cannot be represented with eight crossings and twenty-four non-blank tiles.

We now turn our attention to mosaics with nine crossings. Just as before, we look at all possible placements, up to symmetry, of nine crossings, eliminate any composite knots, unnecessary loops, links and any duplicate layouts that are equivalent to others via simple planar isotopy moves. In the end, we get seven possible layouts, which are shown in Figure 31. Choosing specific crossings for each layout, in order, so that the knots are alternating, we obtain the seven knots 9_8 , 9_{11} , 9_{14} , 9_{17} , 9_{23} , 9_{27} , and 9_{31} , all of which have tile number 24. If we look at all possible choices for non-alternating crossings, the only knot with tile number 24 that arises but did not show up with only eight crossing tiles is the 8_6 knot, whose knot mosaic in the table of knots comes from the layout in Figure 31(a). All other prime knots that arise using non-alternating crossings have been exhibited as a minimally space-efficient mosaic with fewer crossings or fewer non-blank tiles.

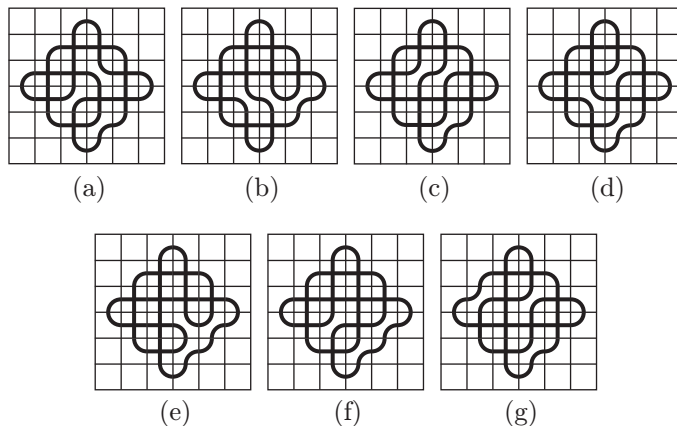


Figure 31: Only possible layouts, after elimination, with nine crossing tiles for a prime knot with tile number 24.

Now we do the same for ten crossings. Again, we observe all possible placements of ten crossings on the third mosaic in Figure 20, and after eliminating any links and duplicate layouts up to reflection, rotation, or equivalencies via simple planar isotopy moves, we end up with five possible layouts, shown in Figure 32.

We begin with Figure 32(a). Choosing specific crossings so that the knot is alternating, we obtain the 10_{116} knot. If we look at all possible choices for non-alternating crossings, the only prime knots that we get with tile number 24 are

the non-alternating knots 10_{124} , 10_{125} , 10_{141} , 10_{143} , 10_{155} , and 10_{159} . We do the same with Figure 32(b) and get the alternating knot 10_{100} . For the non-alternating choices, we get almost all of the same ones we just obtained, but we do not get any new additions to our list of knots. For Figure 32(c), with alternating crossings we get 10_{41} , and with non-alternating crossings we get 9_{19} and 9_{21} as the only new additions to our list. Neither of these came from considering only nine crossings. Now we observe the mosaic in Figure 32(d). By alternating the crossings, we obtain 10_{44} , and by using non-alternating crossings, the only new additions to our list are 9_{12} and 9_{26} . Finally, we end with Figure 32(e). Assigning alternating crossings, we get 10_{85} , and assigning non-alternating crossings, we get 10_{126} , 10_{127} , and 10_{148} .

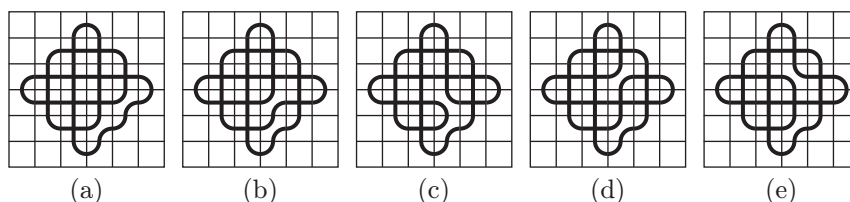


Figure 32: Only possible layouts, after elimination, with ten crossing tiles for a prime knot with tile number 24.

Finally, we can place eleven or twelve crossing tiles into the layout with twenty-four non-blank tiles, but the space-efficient results will always be a link with more than one component. Therefore, no minimally space-efficient prime knot mosaics arise from this consideration. We have considered every possible placement of crossing tiles on the third layout in Figure 20 and have found every possible prime knot with tile number 24 and eight or more crossings, and they are exactly those listed in the theorem. Minimally space-efficient mosaics for all of these knots are given in the table of knots in Section 8. All other prime knots with crossing number at least eight must have tile number larger than 24. \square

We now know precisely which prime knots have tile number less than or equal to 24. Now we determine which prime knots with mosaic number 6 have tile number 27. We do this in the proof of Theorem 23.

Theorem 23. *The only prime knots K with mosaic number $m(K) = 6$ and tile number $t(K) = 27$ are:*

- (a) 8_{15}
- (b) $9_1, 9_2, 9_3, 9_4, 9_7, 9_9, 9_{13}, 9_{24}, 9_{28}, 9_{37}, 9_{46}, 9_{48},$
- (c) $10_1, 10_2, 10_3, 10_4, 10_{12}, 10_{22}, 10_{28}, 10_{34}, 10_{63}, 10_{65}, 10_{66}, 10_{75}, 10_{78}, 10_{140},$
 $10_{142}, 10_{144},$
- (d) $11a_{107}, 11a_{140},$ and $11a_{343}.$

Proof. Similar to what we did in the proof of Theorem 21, we search for all of the prime knots that have tile number 27. We know from Theorem 20 that any prime knot with mosaic number 6 and tile number 27 has a space-efficient mosaic as depicted in the fourth layout of Figure 20.

We simply need to build this layout using the 3×3 building blocks that result from the observations at the beginning of this section and shown again in Figure 33. We then choose specific crossing types for each crossing tile and see what knots we get.

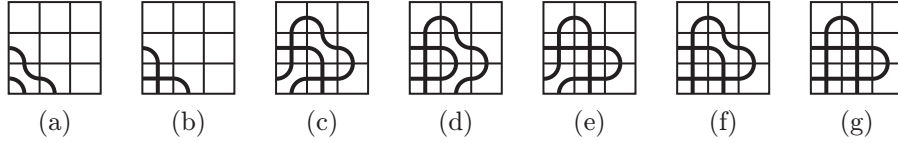


Figure 33: The seven building blocks resulting from the observations at the beginning of this section.

For bookkeeping purposes, we note that the knot 8_{15} has tile number 27, and this is the only knot with crossing number 8 for which we have not previously found the tile number. A minimally space-efficient mosaic for it is included in the table of knots in Section 8. So we now know the tile number for every prime knot with crossing number 8 or less. We now compile a list of the prime knots with nine or more crossings that can fit within this layout of twenty-seven non-blank tiles, omitting those with tile number less than 27. Once this is completed, we will not only know which prime knots have tile number 27, but we will also know how many crossing tiles are necessary for each one.

Before we get started placing crossing tiles, we make a few more simple observations that apply to this particular case and help us reduce the number of possible configurations. Observe that if we place a partially filled building block with no crossing adjacent to the filled building block with two crossing tiles in positions I^3 and I^4 depicted in Figure 33(c), the resulting mosaic will always reduce to a mosaic with tile number 22. The same result holds if the two blocks are not adjacent and one of the adjacent blocks is the filled building block with three crossings depicted in Figure 33(e). The mosaics in Figure 34 exhibit these scenarios. The same result also holds if the partially filled building block with one crossing is combined with two of the filled building blocks with two crossing tiles shown in Figure 33(c). Depending on the placement of these two filled blocks, the result will be equivalent to either Figure 34(a) or Figure 34(b) via a simple planar isotopy move that shifts the crossing in the partially filled block to another block.

First, we consider nine crossing tiles with the above observations in mind, together with the observations at the beginning of this section. Up to symmetry, there are only nine possible configurations of the building blocks after we eliminate the links, duplicate layouts that are equivalent to others via simple planar isotopy

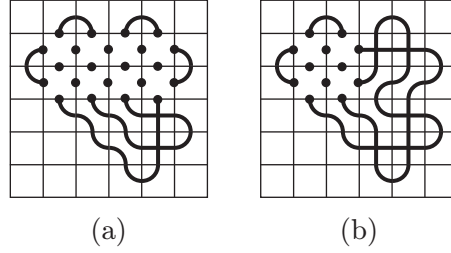


Figure 34: These two mosaics are not minimally space-efficient.

moves, and any mosaics for which the tile number can easily be reduced by a simple planar isotopy move. They are shown in Figure 35. Not all of these will result in distinct knots, and in several cases it is not difficult to see that they will result in the same knot. However, we include all of them here because they differ by more than just symmetries or a simple planar isotopy move.

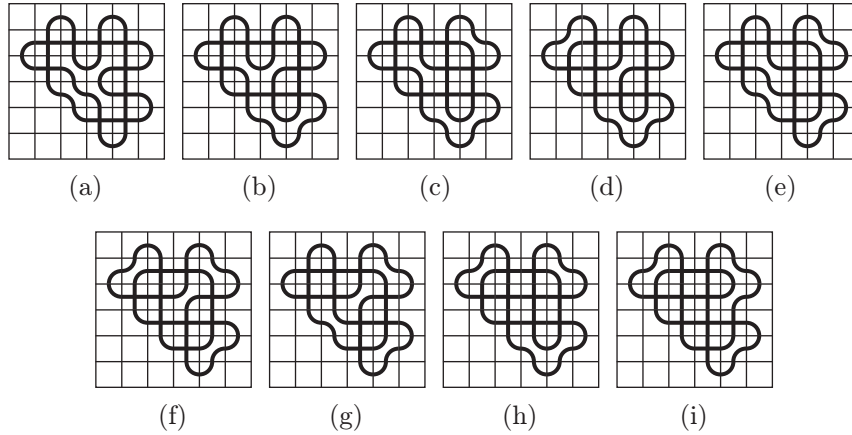


Figure 35: Only possible layouts, after elimination, with nine crossing tiles for a prime knot with tile number 27.

Choosing specific crossings so that the knots are alternating, we obtain only seven distinct knots. The only ones with tile number 27 are Figure 35(a), which gives the 9_1 knot, Figure 35(b), which gives us 9_2 , and Figures 35(h) and (i), which give us 9_{28} . Each of the remaining layouts give knots with tile number less than 27. In particular, Figures 35(c) and (d) are 9_8 , Figures 35(e) and (f) are 9_{17} , and Figure 35(g) is 9_{20} . None of these configurations give non-alternating knots with crossing number 9.

Second, we do the same for ten crossings. Again, we use the building blocks to build all possible configurations of the crossings, and up to symmetry, there are only six possibilities after eliminating any links and duplicate layouts that are equivalent via simple planar isotopy moves. These are shown in Figure 36.

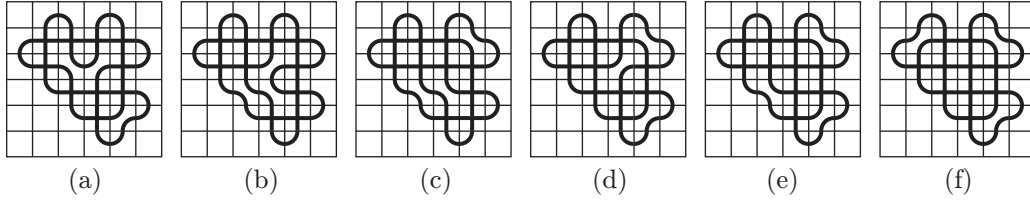


Figure 36: Only possible layouts, after elimination, with ten crossing tiles for a prime knot with tile number 27.

Choosing specific crossings so that the knots are alternating, we obtain only five distinct knots, all of which have tile number 27. In particular, Figure 36(a) becomes the 10_2 knot, Figure 36(b) becomes 10_4 , Figures 36(c) and (d) become 10_{28} , Figure 36(e) becomes 10_{66} , and Figure 36(f) becomes 10_{75} . Choosing non-alternating crossings, we also get some knots with crossing number nine, but we do not obtain any non-alternating knots with crossing number ten. We can get 9_3 from Figure 36(a), 9_4 from Figure 36(b), 9_{13} from Figure 36(c), and 9_{37} , 9_{46} , and 9_{48} from Figure 36(f). All other knots that are obtained by considering non-alternating crossings can be drawn with fewer crossings or a lower tile number.

Third, we consider the case where the mosaic has eleven crossing tiles. In this instance, we end up with the five possible layouts shown in Figure 37, and again, not all of these are distinct. Choosing alternating crossing in each layout results in three distinct knots with crossing number eleven. Figures 37(a) and (b) become $11a_{107}$, Figures 37(c) and (d) become $11a_{140}$, and Figure 37(e) becomes $11a_{343}$. (Note that, for knots with crossing number greater than ten, we are using the Dowker-Thistlethwaite name of the knot.) Choosing non-alternating crossings in each of the layouts results in several knots with crossing number nine or ten. In particular, we can obtain the knots 9_{24} , 10_{63} , 10_{65} , 10_{78} , 10_{140} , 10_{142} , and 10_{144} from Figure 37(a). We can obtain 9_7 , 9_9 , 10_{12} , 10_{22} , and 10_{34} from Figure 37(c). And we can obtain 10_1 and 10_3 from Figure 37(e). All of these are shown in the table of knots in Section 8. All other knots that are obtained by considering non-alternating crossings can be drawn with fewer crossings or a lower tile number.

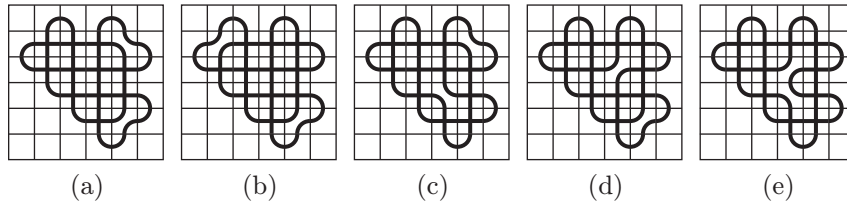


Figure 37: Only possible layouts, after elimination, with eleven crossing tiles for a prime knot with tile number 27.

Finally, if we place twelve or thirteen crossing tiles into the layout with twenty-

seven non-blank tiles, the results will always be a link with more than one component. Therefore, no minimally space-efficient prime knot mosaics arise from this consideration. We have considered every possible placement of nine or more crossing tiles on the fourth layout in Figure 20 and have found every possible prime knot with mosaic number 6 and tile number 27. They are exactly those listed in the theorem. All other prime knots with crossing number at least nine and mosaic number 6 must have minimal mosaic tile number 32. \square

Now we know the tile number for every prime knot with crossing number less than or equal to 8. Theorems 21, 22, and 23 tell us the tile number of some of the prime knots with crossing number 9, 10, and 11. Furthermore, we know that all other prime knots with mosaic number 6 must have minimal mosaic tile number 32 but not necessarily tile number 32. One problem that complicates the next step is that, as of the writing of this paper, we do not know the mosaic number of all prime knots with crossing number 9 or more. That is, we do not know all prime knots with mosaic number 6. For this reason, we need to go through the same process as we did in the preceding proofs to determine which prime knots have mosaic number 6 and minimal mosaic tile number 32. By doing this, we will also be able to determine which prime knots have mosaic number greater than 6. The good news is that this is the final step in determining which prime knots have mosaic number 6 or less and determining the tile number or minimal mosaic tile numbers of all of these.

Theorem 24. *The only prime knots K with mosaic number $m(K) = 6$ and minimal mosaic tile number $t_m(K) = 32$ are:*

- (a) $9_{10}, 9_{16}, 9_{35},$
- (b) $10_{11}, 10_{20}, 10_{21}, 10_{61}, 10_{62}, 10_{64}, 10_{74}, 10_{76}, 10_{77}, 10_{139},$
- (c) $11a_{43}, 11a_{44}, 11a_{46}, 11a_{47}, 11a_{58}, 11a_{59}, 11a_{106}, 11a_{139}, 11a_{165}, 11a_{166}, 11a_{179},$
 $11a_{181}, 11a_{246}, 11a_{247}, 11a_{339}, 11a_{340}, 11a_{341}, 11a_{342}, 11a_{364}, 11a_{367},$
- (d) $11n_{71}, 11n_{72}, 11n_{73}, 11n_{74}, 11n_{75}, 11n_{76}, 11n_{77}, 11n_{78},$
- (e) $12a_{119}, 12a_{165}, 12a_{169}, 12a_{373}, 12a_{376}, 12a_{379}, 12a_{380}, 12a_{444}, 12a_{503}, 12a_{722},$
 $12a_{803}, 12a_{1148}, 12a_{1149}, 12a_{1166},$
- (f) $13a_{1230}, 13a_{1236}, 13a_{1461}, 13a_{4573}$
- (g) $13n_{2399}, 13n_{2400}, 13n_{2401}, 13n_{2402}, 13n_{2403}.$

Proof. We simply go through the same process that we did in the previous proof. We search for all of the prime knots that have mosaic number 6 and minimal mosaic tile number 32. Whatever prime knots that do not show up in this process and that we have not previously determined the tile number for must have mosaic number greater than 6. We know from Theorem 20 that any prime knot with mosaic number

6 and minimal mosaic tile number 32 has a space-efficient mosaic with the fifth and final layout shown in Figure 20.

As we have done several times previously, we use the building blocks to achieve all possible configurations, up to symmetry, of nine or more crossings within this mosaic. For this particular layout, we can only use the filled blocks, not the partially filled blocks. We can eliminate any layouts that do not meet the requirements of the observations, any multi-component links, any duplicate layouts that are equivalent to others via simple planar isotopy moves, and any mosaics for which the tile number can easily be reduced by a simple planar isotopy move.

First, in the case of nine crossings, after we eliminate the unnecessary layouts we end up with only one possibility, and it is shown in Figure 38. However, once we choose specific crossings in an alternating fashion, it is the knot 9_8 , which has tile number 24. Nothing new arises from considering non-alternating crossings either.

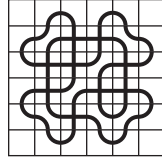


Figure 38: Only possible layout, after elimination, with nine crossing tiles for a prime knot with minimal mosaic tile number 32.

Second, we do the same for ten crossings, and we end up with five possible layouts, shown in Figure 39. Choosing alternating crossings in each one, we again fail to get any prime knots with minimal mosaic tile number 32. Figure 39(a) is 10_1 , Figure 39(b) and (c) are 10_{34} , and Figures 39(d) and (e) are 10_{78} . Nothing new arises from considering non-alternating crossings either.

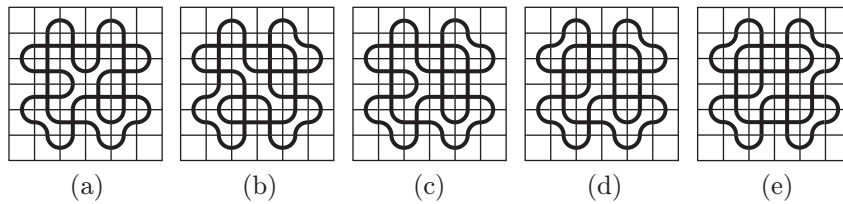


Figure 39: Only possible layouts, after elimination, with ten crossing tiles for a prime knot with minimal mosaic tile number 32.

Third, we consider the case where the mosaic has eleven crossing tiles. In this instance, we end up with the ten possible layouts shown in Figure 40. With alternating crossings, the first layout is $11a_{140}$, which we already known has tile number 27. The remaining layouts, given alternating crossings, lead to six distinct knots with minimal mosaic tile number 32, and with non-alternating crossings we get ten additional knots that have minimal mosaic tile number 32. In particular, Figure 40(b)

with alternating crossings is $11a_{43}$ and with non-alternating crossings can be made into $11n_{71}$, $11n_{72}$, $11n_{73}$, $11n_{74}$, and $11n_{75}$. Figures 40(c) and (d) are $11a_{46}$ when using alternating crossings and can be made into 9_{16} or 10_{77} with non-alternating crossings. Figures 40(e) and (f) are $11a_{59}$ when using alternating crossings and can be made into 10_{20} with non-alternating crossings. Figures 40(g) and (h) are $11a_{179}$ when using alternating crossings and can be made into 9_{10} or 10_{21} with non-alternating crossings. Figure 40(i) with alternating crossings is $11a_{247}$, and Figure 40(j) with alternating crossings is $11a_{367}$. Neither of these last two provide new knots to our list when considering non-alternating crossings.

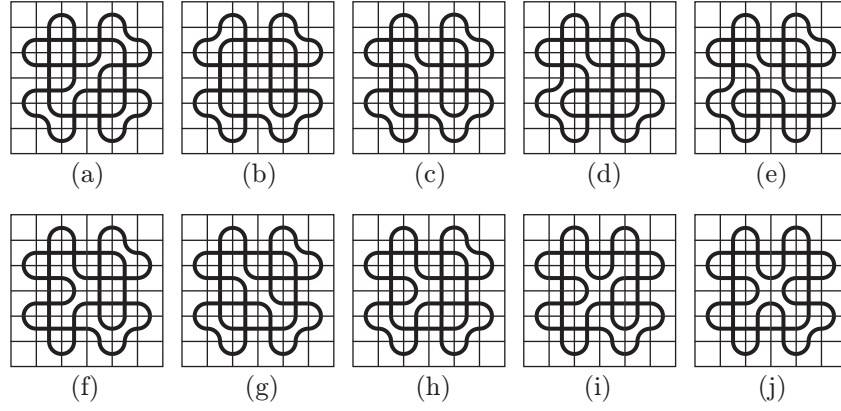


Figure 40: Only possible layouts, after elimination, with eleven crossing tiles for a prime knot with minimal mosaic tile number 32.

Fourth, we consider the possibilities where the mosaic has twelve crossing tiles. In this case, we end up with the seven possible layouts shown in Figure 41. With alternating crossings, these layouts lead to five distinct knots with minimal mosaic tile number 32, and with non-alternating crossings we get thirteen additional knots that have minimal mosaic tile number 32. In particular, Figures 41(a) and (b) with alternating crossings are $12a_{373}$ and with non-alternating crossings can be made into 10_{62} , 10_{64} , 10_{139} , $11a_{106}$, or $11a_{139}$. Figures 41(c) and (d) are $12a_{380}$ when using alternating crossings and can be made into 10_{11} , $11a_{166}$, or $11a_{341}$ with non-alternating crossings. Figure 41(e) is $12a_{503}$ when using alternating crossings and can be made into 9_{35} , 10_{74} , or $11a_{181}$ with non-alternating crossings. Figure 41(f) is $12a_{722}$ when using alternating crossings and can be made into $11a_{364}$ with non-alternating crossings. Figure 41(g) with alternating crossings is $12a_{1149}$ and with non-alternating crossings can be $11a_{342}$.

Fifth, we consider what happens when we place thirteen crossing tiles on the mosaic. In this instance, we end up with the six possible layouts shown in Figure 42. With alternating crossings, the layouts lead to four distinct knots with minimal mosaic tile number 32, and with non-alternating crossings we get twenty-six additional knots that have minimal mosaic tile number 32. In particular, Figure 42(a)

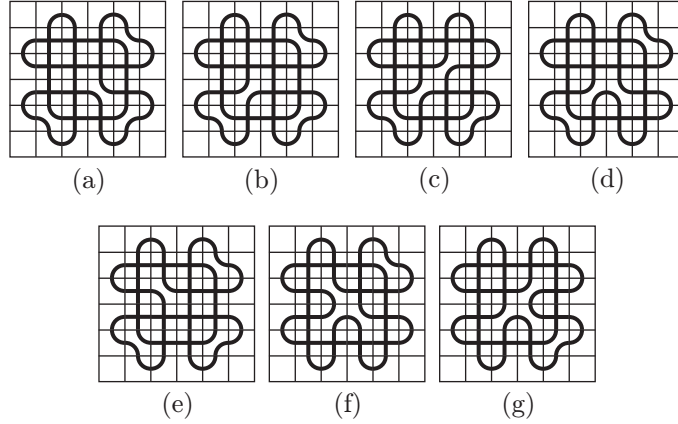


Figure 41: Only possible layouts, after elimination, with twelve crossing tiles for a prime knot with minimal mosaic tile number 32.

with alternating crossings is $13a_{1230}$ and with non-alternating crossings can be made into $11a_{44}$, $11a_{47}$, $11n_{76}$, $11n_{77}$, $11n_{78}$, $12a_{119}$, $13n_{2399}$, $13n_{2400}$, $13n_{2401}$, $13n_{2402}$, or $13n_{2403}$. Figures 42(b) and (c) are $13a_{1236}$ when using alternating crossings and can be made into 10_{61} , 10_{76} , $11a_{58}$, $11a_{165}$, $11a_{340}$, $12a_{165}$, $12a_{376}$, or $12a_{444}$ with non-alternating crossings. Figures 42(d) and (e) are $13a_{1461}$ when using alternating crossings and can be made into $11a_{246}$, $11a_{339}$, $12a_{169}$, $12a_{379}$, or $12a_{1148}$ with non-alternating crossings. Figure 42(f) is $13a_{4573}$ when using alternating crossings and can be made into $12a_{803}$ or $12a_{1166}$ with non-alternating crossings.

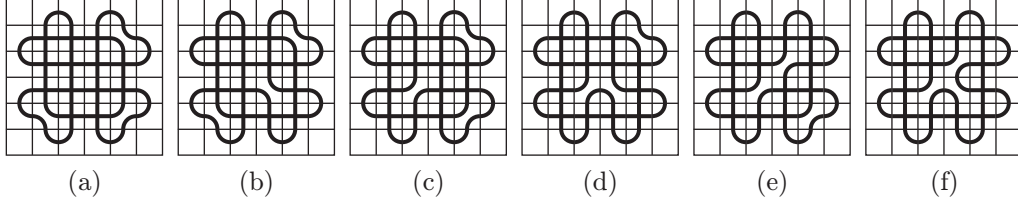


Figure 42: Only possible layouts, after elimination, with thirteen crossing tiles for a prime knot with minimal mosaic tile number 32.

Finally, fourteen (or more) crossing tiles in this layout with thirty-two non-blank tiles requires at least two filled building blocks with four crossings, which is not allowed as it will always be a link with more than one component. Therefore, no minimally space-efficient prime knot mosaics arise from this consideration. We have considered every possible placement of nine or more crossing tiles on the final layout of Figure 20 and have found every possible prime knot with mosaic number 6 and minimal mosaic tile number 32. \square

Because of the work we have completed, we now know every prime knot with

mosaic number 6 or less. We also know the tile number or minimal mosaic tile number of each of these prime knots. In the table of knots in Section 8, we provide minimally space-efficient knot mosaics for all of these. All other prime knots with crossing number 9 or more not listed in Theorems 21, 22, 23, and 24 must have mosaic number 7 or larger.

7 Conjectures and Questions

We take advantage of this final section to look ahead at larger mosaics. We begin with a conjecture about the possible tile numbers of space-efficient 7-mosaics.

Conjecture 27. *If we have a space-efficient 7-mosaic of a prime knot K for which either every column or every row is occupied, then the only possible values for the tile number of the mosaic are 27, 29, 31, 32, 34, 36, 37, 39 and 41.*

Although we have not yet completed a proof of this conjecture, our preliminary investigations using the lemmas in Section 3 led us to these numbers, and the possible layouts for these mosaics are similar to the five possible layouts we found for 6-mosaics. This conjecture also gives additional information about the difference between tile number and minimal mosaic tile number. Notice that some of the possible tile numbers for a space-efficient 7-mosaic are smaller than the tile number 32 for some space-efficient 6-mosaics. It was our investigation of this that helped us realize that tile number and minimal mosaic tile number were not necessarily equal and that we should distinguish between tile number of a knot and minimal mosaic tile number of a knot. We believe keeping these two concepts separate makes the most sense and is the best approach. For our purposes, finding minimally space-efficient mosaics of prime knots, we wanted the mosaics to be minimal. Then, under that constraint, we find the smallest possible tile number of the knot.

Because of this conjecture, we have the following question about the prime knots listed in Theorem 24 with minimal mosaic tile number 32. The conjecture implies that many of them could have a smaller tile number of 27, 29, or 31.

Question. What is the tile number of the prime knots K with mosaic number $m(K) = 6$ and minimal mosaic tile number $t_m(K) = 32$?

Our work with prime knots naturally leads to a question about n -mosaics of composite knots and multi-component links. In particular, we know the bounds for the tile number of prime knots from Theorem 12. What about other knots and links?

Question. What are the bounds for the tile numbers of links and composite knots with mosaic number n ?

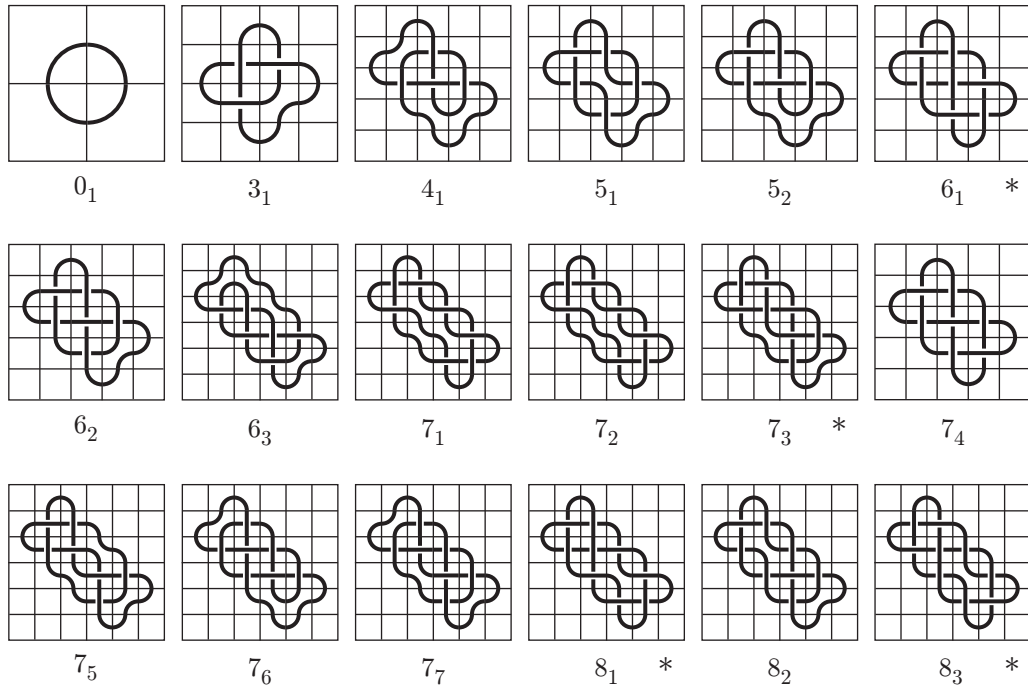
If one looks closely at all of the minimally space-efficient mosaics provided in the table of mosaics in Section 8, one will notice that the line segment tiles T_5 and

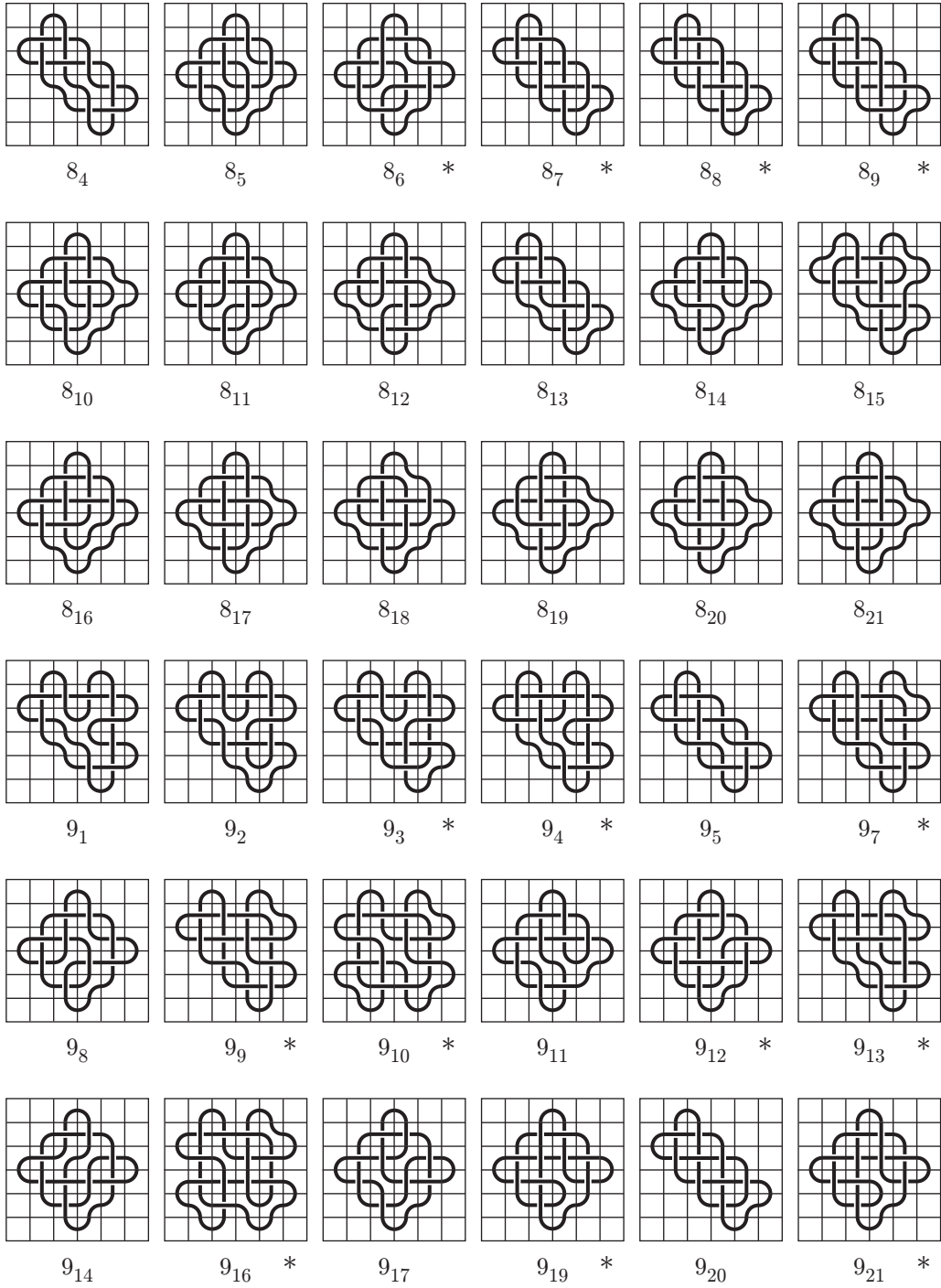
T_6 are not used. In fact, in every case that one of these tiles was used with regard to the lemmas in Section 3, the mosaic was not space-efficient or the line segments could be changed to arcs by pushing them into adjacent tiles or collapsing them. That is, for space-efficient n -mosaics with $n \leq 6$, the line segment tiles T_5 or T_6 are not necessary. We believe that this will always be the case.

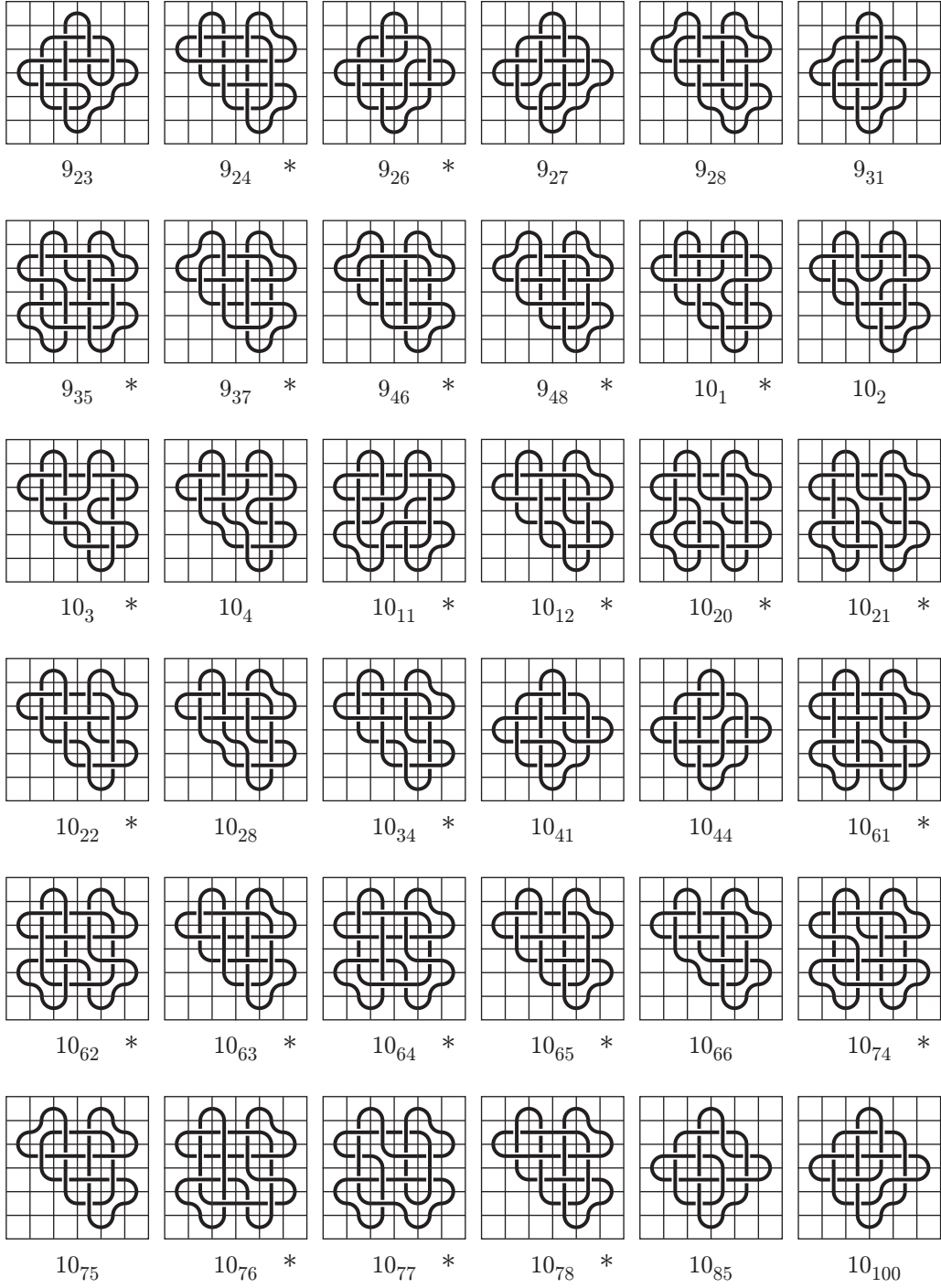
Conjecture 28. *The line segment tiles T_5 or T_6 are not necessary for creating knot mosaics.*

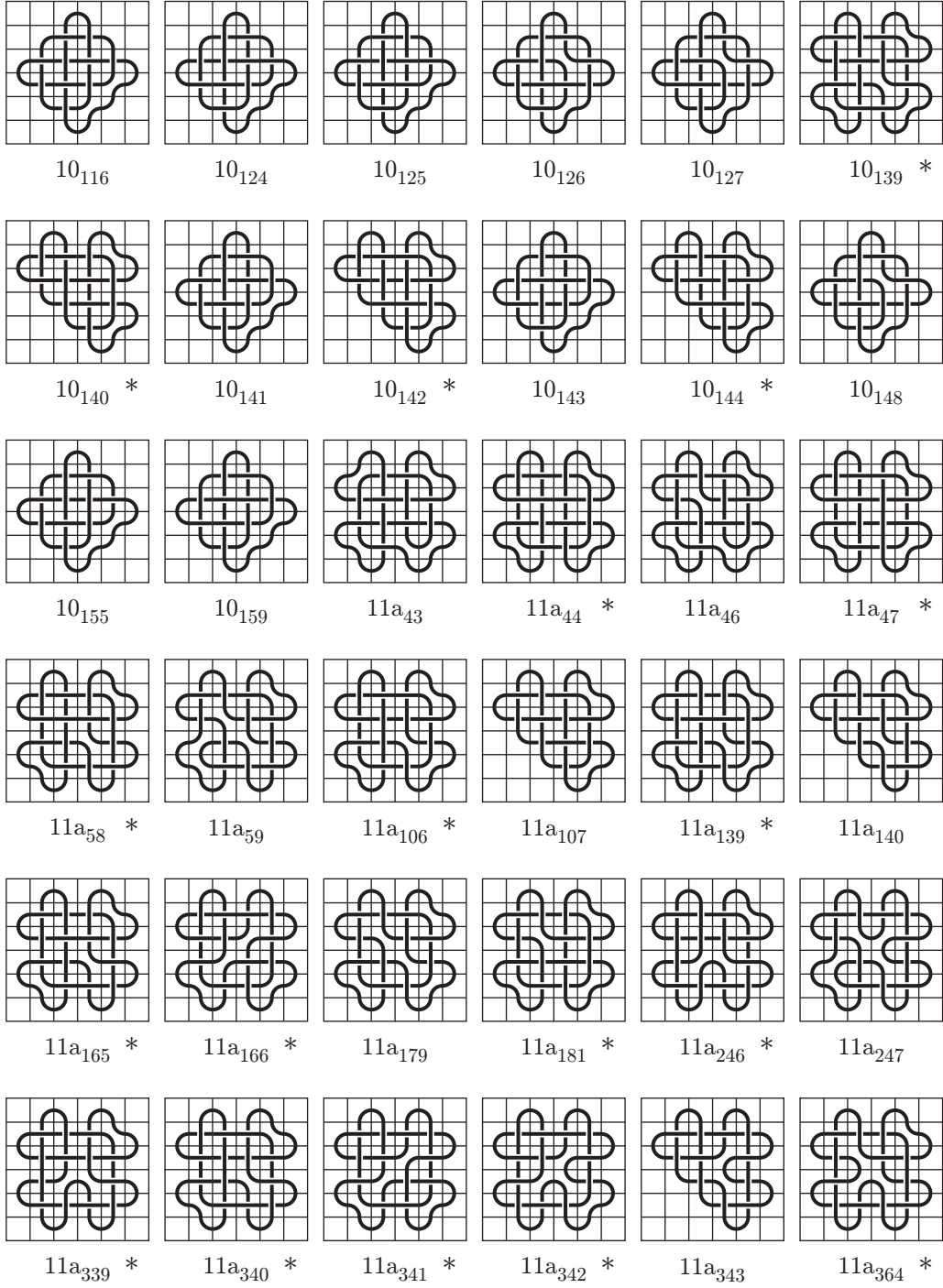
8 Table of Prime Knots

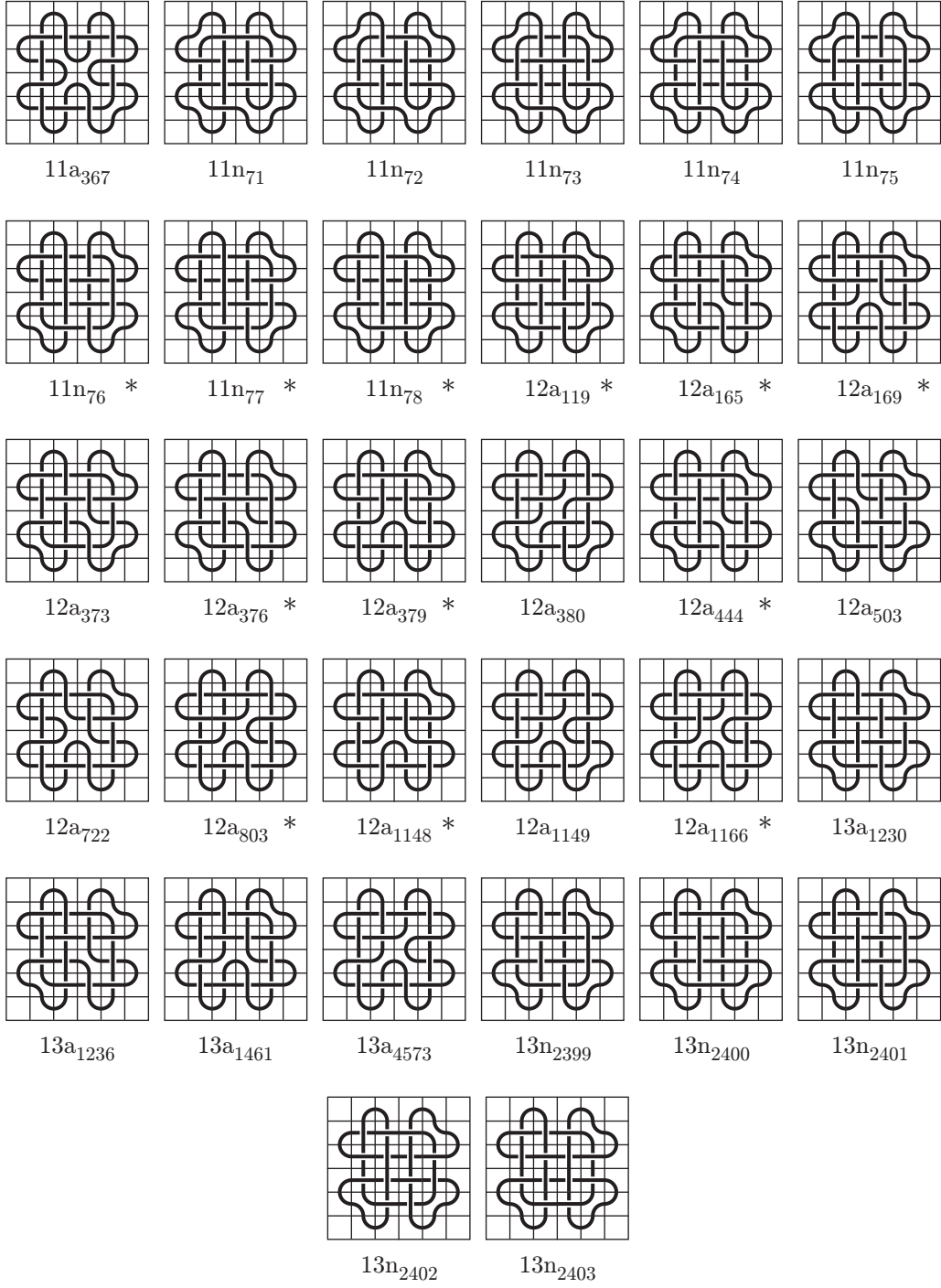
We include a table of knots below, with an example of a minimally space-efficient knot mosaic for each prime knot for which we know the minimal mosaic tile number. For each knot mosaic, both the mosaic number and minimal mosaic tile number are realized, but the crossing number may not be realized. The tile number of the knot is realized in each mosaic unless the tile number of the mosaic is 32. If the knot mosaic is marked with an asterisk (*) then the given mosaic has more crossing tiles than the crossing number for the represented knot, but it is the minimum number of crossing tiles needed in order for the minimal mosaic tile number to be realized.











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